Overconvergent de Rham-Witt Cohomology for Algebraic Stacks

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Slides available at http://www.mathcs.emory.edu/~dzb/slides/

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Weil Conjectures

Throughout, $p$ is a prime and $q = p^n$.

**Definition**

The **zeta function** of a variety $X$ over $\mathbb{F}_q$ is the series

$$\zeta_X(T) = \exp \left( \sum_{n=1}^{\infty} \frac{\#X(\mathbb{F}_{q^n}) T^n}{n} \right).$$

**Rationality**: For $X$ smooth and proper of dimension $d$

$$\zeta_X(T) = \frac{P_1(T) \cdots P_{2d-1}(T)}{P_0(T) \cdots P_{2d}(T)}$$

**Cohomological description**: For any Weil cohomology $H^i$,

$$P_i(T) = \det(1 - T \text{Frob}_q, H^i(X)).$$
Weil Conjectures

Definition

The **zeta function** of a variety $X$ over $\mathbb{F}_q$ is the series

$$
\zeta_X(T) = \exp \left( \sum_{n=1}^{\infty} \#X(\mathbb{F}_{q^n}) \frac{T^n}{n} \right).
$$

**Consequences for point counting:**

$$
\#X(\mathbb{F}_{q^n}) = \sum_{r=0}^{2d} (-1)^r \sum_{i=1}^{b_r} \alpha_{i,r}^n
$$

**Riemann hypothesis** (Deligne):

$$
P_i(T) \in 1 + T\mathbb{Z}[T], \text{ and the } \mathbb{C}\text{-roots } \alpha_{i,r} \text{ of } P_i(T) \text{ have norm } q^{i/2}.
$$
Fact
For a prime $p$, the condition that two proper varieties $X$ and $X'$ over $\mathbb{Z}_p$ with good reduction at $p$ have the same reduction at $p$ implies that their Betti numbers agree.

Explanation

$$H^i_{\text{cris}}(X_p/\mathbb{Z}_p) \cong H^i_{\text{dR}}(X, \mathbb{Z}_p)$$
1. The **Newton Polygon** of $X$ is the lower convex hull of $(i, v_p(a_i))$.

2. The **Hodge Polygon** of $X$ is the polygon whose slope $i$ segment has width

   $$h^{i, \text{dim}(X) - i} := H^i(X, \Omega_{X}^{\text{dim}(X) - i}).$$

3. **Example**: $E$ supersingular elliptic curve.

   ![Newton and Hodge Polygons](Newton above Hodge)

   - **Newton**
   - **Hodge**
Modern:

(Étale) \[ H^i_{\text{et}}(X, \mathbb{Q}_\ell) \]

(Crystalline) \[ H^i_{\text{cris}}(X/W) \]

(Rigid/overconvergent) \[ H^i_{\text{rig}}(X) \]

Variants, preludes, and complements:

(Monsky-Washnitzer) \[ H^i_{\text{MW}}(X) \]

(de Rham-Witt) \[ H^i(X, W\Omega_X^\bullet) \]

(overconvergent dRW) \[ H^i(X, W^\dagger\Omega_X^\bullet) \]
Let $X$ be a smooth variety over $\mathbb{F}_q$.

**Theorem (Illusie, 1975)**

There exists a complex $\mathcal{W}\Omega_X^\bullet$ of sheaves on the Zariski site of $X$ whose (hyper)cohomology computes the crystalline cohomology of $X$.

**1 Main points**

1. Sheaf cohomology on **Zariski** rather than the **crystalline** site.
2. Complex is independent of choices (compare with Monsky-Washnitzer).
3. Somewhat explicit.
de Rham-Witt

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Applications - easy proofs of

1. Finite generation.
2. Torsion-free case of Newton above Hodge

Generalizations

1. Langer-Zink (relative case).
2. Hesselholt (big Witt vectors).
Definition of $\Omega_X^\bullet$

1. It is a particular quotient of $\Omega^\bullet_{W(X)/W(F_p)}$.
2. Recall: if $A$ is a perfect ring of char $p$, $W(A) = \prod A \ni \sum a_i p^i \ (a_i \in A)$

What is $W(k[x])$?

1. $W(k[x]) \subset W(k[x]^{\text{perf}}) \ni \sum_{k \in \mathbb{Z}[1/p]} a_k x^k$
2. $f = \sum_{k \in \mathbb{Z}[1/p]} a_k x^k \in W(k[x])$ if $f$ is $V$-adically convergent, i.e.,
3. $v_p(a_k) \geq 0$.
4. (i.e., $V(x) = px^{\frac{1}{p}} \in W(k[x])$, but $x^{\frac{1}{p}} \not\in W(k[x])$.)
For $X$ a general scheme (or stack), one can glue this construction.

$\mathcal{W}\Omega_X^\bullet$ is an initial $V$-pro-complex.
Theorem (Davis, Langer, and Zink)

There is a subcomplex

$$W^\dagger \Omega_X \subset W\Omega_X$$

such that if $X$ is a smooth scheme, $H^i(X, W^\dagger \Omega_X) \otimes \mathbb{Q} \cong H^i_{\text{rig}}(X)$.

Note well:

1. Left hand side is Zariski cohomology.
2. (Right hand side is cohomology of a complex on an associated rigid space.)
3. The complex $W^\dagger \Omega_X$ is independent of choices and functorial.
\[ \# \mathcal{B} \mathbb{G}_m(\mathbb{F}_p) = \sum_{x \in |\mathcal{B} \mathbb{G}_m(\mathbb{F}_p)|} \frac{1}{\# \text{Aut}_x(\mathbb{F}_p)} = \frac{1}{p - 1} \]

\[ = \sum_{i=-\infty}^{\infty} (-1)^i \text{Tr Frob}_{c, \text{ét}} H^i_{c, \text{ét}}(\mathcal{B} \mathbb{G}_m, \overline{\mathbb{Q}}_\ell) \]

\[ = \sum_{i=1}^{\infty} (-1)^2 p^{-i} = \frac{1}{p} + \frac{1}{p^2} + \frac{1}{p^3} + \cdots \]

**Example**

Étale cohomology and Weil conjectures for stacks are used in Ngô's proof of the fundamental lemma.
1 Book by Martin Olsson “Crystalline cohomology of algebraic stacks and Hyodo-Kato cohomology”.

2 Main application – new proof of $C_{st}$ conjecture in $p$-adic Hodge theory.

3 Key insight –

$$H^i_{\log\text{-}\text{cris}}(X, M) \cong H^i_{\text{cris}}(\text{Log}(X, M)).$$

4 One technical ingredient – generalizations of de Rham-Witt complex to stacks. (Needed, e.g., to prove finiteness.)
Original motivation: **Geometric Langlands** for $\text{GL}_n(\mathbb{F}_p(C))$:

1. Lafforgue constructs a ‘compactified moduli stack of shtukas’ $\mathcal{X}$ (actually a compactification of a stratification of a moduli stack of shtukas).

2. The $\ell$-adic étale cohomology of étale sheaves on $\mathcal{X}$ **realize a Langlands correspondence** between certain **Galois** and **automorphic** representations.

Other motivation: applications to **log-rigid** cohomology.
Rigid cohomology for stacks

Theorem (ZB, thesis)

1. **Definition** of rigid cohomology for stacks (via le Stum’s overconvergent site)
2. Define variants with **supports in a closed subscheme**,
3. show they **agree** with the classical constructions.
4. **Cohomological descent** on the overconvergent site.
In progress (ZB)

1. Duality.
2. Compactly supported cohomology.
3. Full Weil formalism.
4. Applications.
Main theorems

**Theorem (Davis-ZB; in preparation)**

Let $\mathcal{X}$ be a smooth Artin stack of finite type over $\mathbb{F}_q$. Then there exists a functorial complex $W^\dagger \Omega^\bullet_{\mathcal{X}}$ whose cohomology agrees with the rigid cohomology of $\mathcal{X}$.

**Theorem (in preparation)**

Let $X$ be a smooth affine scheme over $\mathbb{F}_q$. Then the étale cohomology $H^i_{\text{ét}}(X, W^\dagger \Omega^j_{\mathcal{X}}) = 0$ for $i > 0$.

**Theorem (Accepted; MRL)**

Integral $MW$-cohomology agrees with overconvergent cohomology (for $i < p$).
Main technical details

Remark

In the classical case, once can write

$$W^i \Omega = \lim_{n} W^i_n \Omega.$$  

The sheaves $W^i_n \Omega$ are coherent.

Remark

In the overconvergent case,

$$W^{\dagger}^i \Omega = \lim_{\epsilon} W^\epsilon \Omega^i.$$
Main technical details

Tools used in the proof

1. Limit Čech cohomology.
2. Topological (in the Grothendieck sense) unwinding lemmas.
4. (Stein property)
   \[ U \subset X \text{ and } \mathcal{O}(U) = \mathcal{O}(X) \Rightarrow W^\dagger \Omega^i(U) \cong W^\dagger \Omega^i(X) \]
5. Nisnevish Devissage.
6. Brutal direct computations: need surjectivity of
   \[ W^\dagger \Omega^i(U) \oplus W^\dagger \Omega^i(X') \rightarrow W^\dagger \Omega^i(X) \]
   for a standard Nisnevich cover \( U \coprod X' \rightarrow X \).