Overconvergent de Rham-Witt Cohomology for Algebraic Stacks

David Zureick-Brown (Emory University)
Christopher Davis (UC Irvine)

Slides available at http://www.mathcs.emory.edu/~dzb/slides/

2013 Joint Math Meetings
Special Session on Witt Vectors, Liftings and Decent
San Diego, CA

January 10, 2013
Throughout, $p$ is a prime and $q = p^n$.

**Definition**

The **zeta function** of a variety $X$ over $\mathbb{F}_q$ is the series

$$\zeta_X(T) = \exp \left( \sum_{n=1}^{\infty} \frac{\# X(\mathbb{F}_{q^n})}{n} T^n \right).$$

**Rationality**: For $X$ smooth and proper of dimension $d$

$$\zeta_X(T) = \frac{P_1(T) \cdots P_{2d-1}(T)}{P_0(T) \cdots P_{2d}(T)}$$

**Cohomological description**: For any Weil cohomology $H^i$,

$$P_i(T) = \det(1 - T \text{Frob}_q, H^i(X)).$$
Weil Conjectures

Definition

The **zeta function** of a variety $X$ over $\mathbb{F}_q$ is the series

$$\zeta_X(T) = \exp \left( \sum_{n=1}^{\infty} \#X(\mathbb{F}_{q^n}) \frac{T^n}{n} \right).$$

**Consequences for point counting:**

$$\#X(\mathbb{F}_{q^n}) = \sum_{r=0}^{2d} (-1)^r \sum_{i=1}^{b_r} \alpha_{i,r}^n.$$ 

**Riemann hypothesis** (Deligne):

$$P_i(T) \in 1 + T\mathbb{Z}[T],$$ and the $\mathbb{C}$-roots $\alpha_{i,r}$ of $P_i(T)$ have norm $q^{i/2}$. 

David Zureick-Brown (Emory)
Fact
For a prime $p$, the condition that two proper varieties $X$ and $X'$ over $\mathbb{Z}_p$ with good reduction at $p$ have the same reduction at $p$ implies that their Betti numbers agree.

Explanation
$$H^i_{\text{cris}}(X_p/\mathbb{Z}_p) \cong H^i_{\text{dR}}(X, \mathbb{Z}_p)$$
1. The **Newton Polygon** of $X$ is the lower convex hull of $(i, \nu_p(a_i))$.

2. The **Hodge Polygon** of $X$ is the polygon whose slope $i$ segment has width

$$h^{i, \dim(X)} := H^i(X, \Omega^{\dim(X)-i}_X).$$

3. **Example**: $E$ supersingular elliptic curve.
Weil cohomologies

Modern:

(Étale) \( H^i_{\text{et}}(X, \mathbb{Q}_\ell) \)

(Crystalline) \( H^i_{\text{cris}}(X/W) \)

(Rigid/overconvergent) \( H^i_{\text{rig}}(X) \)

Variants, preludes, and complements:

(Monsky-Washnitzer) \( H^i_{\text{MW}}(X) \)

(de Rham-Witt) \( H^i(X, W\Omega^\bullet_X) \)

(overconvergent dRW) \( H^i(X, W^\dagger\Omega^\bullet_X) \)
Let $X$ be a smooth variety over $\mathbb{F}_q$.

**Theorem (Illusie, 1975)**

There exists a complex $\mathcal{W}\Omega^\bullet_X$ of sheaves on the Zariski site of $X$ whose (hyper)cohomology computes the crystalline cohomology of $X$.

**Main points**

1. Sheaf cohomology on **Zariski** rather than the **crystalline** site.
2. Complex is independent of choices (compare with Monsky-Washnitzer).
3. Somewhat explicit.
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**Theorem (Illusie, 1975)**

There exists a complex $\mathcal{W}\Omega^\bullet_X$ of sheaves on the Zariski site of $X$ whose (hyper)cohomology computes the crystalline cohomology of $X$.

1. Applications - easy proofs of
   1. Finite generation.
   2. Torsion-free case of Newton above Hodge
2. Generalizations
   1. Langer-Zink (relative case).
   2. Hesselholt (big Witt vectors).
Definition of $W\Omega_{\mathcal{X}}$

1. It is a particular quotient of $\Omega_{W(X)/W(F_p)}$.

2. Recall: if $A$ is a perfect ring of char $p$,

$$W(A) = \prod A \ni \sum a_i p^i \quad (a_i \in A)$$

What is $W(k[x])$?

1. $W(k[x]) \subset W(k[x]^{\text{perf}}) \ni \sum_{k \in \mathbb{Z}[1/p]} a_k x^k$

2. $f = \sum_{k \in \mathbb{Z}[1/p]} a_k x^k \in W(k[x])$ if $f$ is $V$-adically convergent, i.e.,

3. $v_p(a_k k) \geq 0$.

4. (i.e., $V(x) = px^{\frac{1}{p}} \in W(k[x])$, but $x^{\frac{1}{p}} \not\in W(k[x])$.)
Definition of $\mathcal{W}\Omega^\bullet_X$

1. For $X$ a general scheme (or stack), one can glue this construction.
2. $\mathcal{W}\Omega^\bullet_X$ is an initial $V$-pro-complex.
Theorem (Davis, Langer, and Zink)

There is a subcomplex

\[ W^\dagger \Omega_X^\bullet \subset W\Omega_X^\bullet \]

such that if \( X \) is a smooth scheme, \( H^i(X, W^\dagger \Omega_X^\bullet) \otimes \mathbb{Q} \cong H^i_{\text{rig}}(X) \).

Note well:

1. Left hand side is Zariski cohomology.
2. (Right hand side is cohomology of a complex on an associated rigid space.)
3. The complex \( W^\dagger \Omega_X^\bullet \) is independent of choices and functorial.
Étale Cohomology for stacks

\[
\# B \mathbb{G}_m(\mathbb{F}_p) = \sum_{x \in |B \mathbb{G}_m(\mathbb{F}_p)|} \frac{1}{\# \text{Aut}_x(\mathbb{F}_p)} = \frac{1}{p - 1}
\]

\[
= \sum_{i=-\infty}^{i=\infty} (-1)^i \text{Tr Frob} H^i_{c,\text{ét}}(B \mathbb{G}_m, \overline{\mathbb{Q}_\ell})
\]

\[
= \sum_{i=1}^{\infty} (-1)^2 p^{-i} = \frac{1}{p} + \frac{1}{p^2} + \frac{1}{p^3} + \cdots
\]

Example

Étale cohomology and Weil conjectures for stacks are used in Ngô’s proof of the fundamental lemma.

2. **Main application** – new proof of $C_{st}$ conjecture in $p$-adic Hodge theory.

3. **Key insight** –

   $$H^i_{\text{log-cris}}(X, M) \cong H^i_{\text{cris}}(\text{Log}(X, M)).$$

4. One technical ingredient – generalizations of de Rham-Witt complex to stacks. (Needed, e.g., to prove finiteness.)
Rigid cohomology for stacks

Original motivation: **Geometric Langlands** for $\text{GL}_n(\mathbb{F}_p(C))$: 

1. Lafforgue constructs a ‘compactified moduli stack of shtukas’ $\mathcal{X}$ (actually a compactification of a stratification of a moduli stack of shtukas).

2. The $\ell$-adic étale cohomology of étale sheaves on $\mathcal{X}$ realize a **Langlands correspondence** between certain **Galois** and **automorphic** representations.

Other motivation: applications to **log-rigid** cohomology.
Rigid cohomology for stacks

Definition of rigid cohomology for stacks (via le Stum’s overconvergent site)

Define variants with supports in a closed subscheme,

show they agree with the classical constructions.

Cohomological descent on the overconvergent site.
In progress

1. Duality.
2. Compactly supported cohomology.
3. Full Weil formalism.
4. Applications.
Main theorem

Theorem (Davis-ZB)

Let $\mathcal{X}$ be a smooth Artin stack of finite type over $\mathbb{F}_q$. Then there exists a functorial complex $W^\dagger \Omega^\bullet_{\mathcal{X}}$ whose cohomology agrees with the rigid cohomology of $\mathcal{X}$.

Theorem

Let $X$ be a smooth affine scheme over $\mathbb{F}_q$. Then the etale cohomology $H^i_{\text{ét}}(X, W^\dagger \Omega^j_X) = 0$ for $i > 0$. 
Main technical detail

In the classical case, once can write

$$\mathcal{W} \Omega^i = \lim_{\leftarrow n} \mathcal{W}_n \Omega^i;$$

the sheaves $\mathcal{W}_n \Omega^i$ are coherent. In the overconvergent case,

$$\mathcal{W}^\dagger \Omega^i = \lim_{\rightarrow \epsilon} \mathcal{W}^\epsilon \Omega^i.$$

Tools used in the proof

1. Limit Čech cohomology.
2. Topological (in the Grothendieck sense) unwinding lemmas.
4. Brutal direct computations.