Rational points on curves and tropical geometry.

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Slides available at http://www.mathcs.emory.edu/~dzb/slides/

Specialization of Linear Series for Algebraic and Tropical Curves
BIRS

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Faltings’ theorem

Theorem (Faltings)

Let $X$ be a smooth curve over $\mathbb{Q}$ with genus at least 2. Then $X(\mathbb{Q})$ is finite.

Example

For $g \geq 2$, $y^2 = x^{2g+1} + 1$ has only finitely many solutions with $x, y \in \mathbb{Q}$. 

Uniformity

Problem
1. Given $X$, compute $X(\mathbb{Q})$ exactly.
2. Compute bounds on $\#X(\mathbb{Q})$.

Conjecture (Uniformity)
There exists a constant $N(g)$ such that every smooth curve of genus $g$ over $\mathbb{Q}$ has at most $N(g)$ rational points.

Theorem (Caporaso, Harris, Mazur)
*Lang's conjecture* $\Rightarrow$ uniformity.
Coleman’s bound

**Theorem (Coleman)**

Let $X$ be a curve of genus $g$ and let $r = \text{rank}_\mathbb{Z} \text{Jac}_X(\mathbb{Q})$. Suppose $p > 2g$ is a prime of good reduction. Suppose $r < g$. Then

$$\#X(\mathbb{Q}) \leq \#X(\mathbb{F}_p) + 2g - 2.$$ 

**Remark**

1. A modified statement holds for $p \leq 2g$ or for $K \neq \mathbb{Q}$.
2. Note: this does not prove uniformity (since the first good $p$ might be large).
Let $X$ be a curve of genus $g$ and let $r = \text{rank}_\mathbb{Z} \text{Jac}_X(\mathbb{Q})$. Suppose $p > 2g$ is a prime of good reduction. Suppose $r < g$. Then

$$\# X(\mathbb{Q}) \leq \# X(\mathbb{F}_p) + 2r.$$
Theorem (Lorenzini-Tucker, McCallum-Poonen)

Let $X$ be a curve of genus $g$ and let $r = \text{rank}_{\mathbb{Z}} \text{Jac}_X(\mathbb{Q})$. Suppose $p > 2g$ is a prime. Suppose $r < g$.

Let $\mathcal{X}$ be a regular proper model of $X$. Then

$$\#X(\mathbb{Q}) \leq \#\mathcal{X}^{\text{sm}}(\mathbb{F}_p) + 2g - 2.$$ 

Remark

A recent improvement due to Stoll gives a uniform bound if $r \leq g - 3$ and $X$ is hyperelliptic.
Theorem (Katz-ZB)

Let $X$ be a curve of genus $g$ and let $r = \text{rank}_\mathbb{Z} \text{Jac}_X(\mathbb{Q})$. Suppose $p > 2g$ is a prime. Let $\mathcal{X}$ be a regular proper model of $X$. Suppose $r < g$. Then

$$#X(\mathbb{Q}) \leq #\mathcal{X}^\text{sm}(\mathbb{F}_p) + 2r.$$
Example (hyperelliptic curve with cuspidal reduction)

\[-2 \cdot 11 \cdot 19 \cdot 173 \cdot y^2 = (x - 50)(x - 9)(x - 3)(x + 13)(x^3 + 2x^2 + 3x + 4)\]
\[= x(x + 1)(x + 2)(x + 3)(x + 4)^3 \mod 5.\]

Analysis

1. \(X(\mathbb{Q})\) contains
   \[\{\infty, (50, 0), (9, 0), (3, 0), (-13, 0), (25, 20247920), (25, -20247920)\}\]

2. \(\#X_{\text{sm}}^5(\mathbb{F}_5) = 5\)

3. \(7 \leq \#X(\mathbb{Q}) \leq \#X_{\text{sm}}^5(\mathbb{F}_5) + 2 \cdot 1 = 7\)

This determines \(X(\mathbb{Q})\).
\[
y^2 = x^6 + 5 = x^6 \mod 5.
\]

**Analysis**

1. \(X(\mathbb{Q}) \supset \{\infty^+, \infty^-\}\)
2. \(\mathcal{X}^\text{sm}(\mathbb{F}_5) = \{\infty^+, \infty^-, \pm(1, \pm 1), \pm(2, \pm 2^3), \pm(3, \pm 3^3), \pm(4, \pm 4^3)\}\)
3. \(2 \leq \#X(\mathbb{Q}) \leq \#\mathcal{X}_5^\text{sm}(\mathbb{F}_5) + 2 \cdot 1 = 20\)
Models \((\mathcal{X}/\mathbb{Z}_p)\)

\[
y^2 = x^6 + 5
\]

\[
= x^6 \mod 5.
\]

Note: no \(\mathbb{Z}_p\)-point can reduce to \((0, 0)\).
Models – not regular

\[ y^2 = x^6 + 5^2 \]

\[ = x^6 \mod 5 \]

Now: \((0, 5)\) reduces to \((0, 0)\).
Models – not regular (blow up)

\[ y^2 = x^6 + 5^2 = x^6 \mod 5 \]

Blow up.
Models – semistable example

\[ y^2 = (x(x - 1)(x - 2))^3 + 5 \]
\[ = x^6 \mod 5. \]

Note: no point can reduce to \((0, 0)\). Local equation looks like \(xy = 5\)
Models – semistable example (not regular)

\[ y^2 = (x(x - 1)(x - 2))^3 + 5^4 \]
\[ = x^6 \mod 5 \]

Now: \((0, 5^2)\) reduces to \((0, 0)\). Local equation looks like \(xy = 5^4\)
Models – semistable example

\[ y^2 = (x(x - 1)(x - 2))^3 + 5^4 \]
\[ = x^6 \mod 5 \]

Blow up. Local equation looks like \( xy = 5^3 \)
Models – semistable example (regular at (0,0))

\[ y^2 = (x(x - 1)(x - 2))^3 + 5^4 \]

\[ = x^6 \mod 5 \]

Blow up. Local equation looks like \( xy = 5 \)
Theorem (Katz-ZB)

Let $X$ be a curve of genus $g$ and let $r = \text{rank}_\mathbb{Z} \text{Jac}_X(\mathbb{Q})$. Suppose $p > 2g$ is a prime. Let $\tilde{X}$ be a regular proper model of $X$. Suppose $r < g$. Then

$$\#X(\mathbb{Q}) \leq \#\tilde{X}^{\text{sm}}(\mathbb{F}_p) + 2r.$$
Chabauty’s method

\textbf{(p-adic integration)} There exists $V \subset H^0(X_{\mathbb{Q}_p}, \Omega^1_X)$ with $\dim_{\mathbb{Q}_p} V \geq g - r$ such that,

\[
\int_P^Q \omega = 0 \quad \forall P, Q \in X(\mathbb{Q}), \omega \in V
\]

\textbf{(Coleman, via Newton Polygons)} Number of zeroes in a residue disc $D_P$ is $\leq 1 + n_P$, where $n_P = \# (\text{div } \omega \cap D_P)$

\textbf{(Riemann-Roch)} $\sum n_P = 2g - 2$.

\textbf{(Coleman’s bound)} $\sum_{P \in X(\mathbb{F}_p)} (1 + n_P) = \# X(\mathbb{F}_p) + 2g - 2$. 
Example (from McCallum-Poonen’s survey paper)

Example

\[ X : y^2 = x^6 + 8x^5 + 22x^4 + 22x^3 + 5x^2 + 6x + 1 \]

1. Points reducing to \( \tilde{Q} = (0, 1) \) are given by

\[
\begin{align*}
    x &= p \cdot t, \text{ where } t \in \mathbb{Z}_p \\
    y &= \sqrt{x^6 + 8x^5 + 22x^4 + 22x^3 + 5x^2 + 6x + 1} = 1 + x^2 + \cdots
\end{align*}
\]

2. \[
\int_{(0,1)}^{P_t} \frac{xdx}{y} = \int_{0}^{t} (x - x^3 + \cdots)dx
\]
(Coleman, via Newton Polygons) Number of zeroes of $\int \omega$ in a residue class $D_P$ is $\leq 1 + n_P$, where $n_P = \# (\text{div} \omega \cap D_P)$

Let $\tilde{n}_P = \min_{\omega \in V} \# (\text{div} \omega \cap D_P)$

(2 examples) $r \leq g - 2$, $\omega_1, \omega_2 \in V$

(Stoll’s bound) $\sum \tilde{n}_P \leq 2r$. (Recall $\dim_{\mathbb{Q}_p} V \geq g - r$)
Stoll’s bound – proof \( (D = \sum \tilde{n}_P P) \)

(Wanted)

\[
\dim H^0(X_{\mathbb{F}_p}, K - D) \geq g - r \Rightarrow \deg D \leq 2r
\]

(Clifford)

\[
H^0(X_{\mathbb{F}_p}, K - D') \neq 0 \Rightarrow \dim H^0(X_{\mathbb{F}_p}, D') \leq \frac{1}{2} \deg D' + 1
\]

(D’ = K – D)

\[
\dim H^0(X_{\mathbb{F}_p}, K - D) \leq \frac{1}{2} \deg(K - D) + 1
\]

(Assumption)

\[
g - r \leq \dim H^0(X_{\mathbb{F}_p}, K - D)
\]

(Recall \( \dim_{\mathbb{Q}_p} V \geq g - r \))
Complications when $X_{\mathbb{F}_p}$ is singular

1. $\omega \in H^0(X, \Omega)$ may vanish along components of $X_{\mathbb{F}_p}$;
2. i.e. $H^0(X_{\mathbb{F}_p}, K - D) \neq 0 \not\Rightarrow D$ is special;
3. $\text{rank}(K - D) \neq \dim H^0(X_{\mathbb{F}_p}, K - D) - 1$

Summary

The relationship between $\dim H^0(X_{\mathbb{F}_p}, K - D)$ and $\deg D$ is less transparent and does not follow from geometric techniques.
Definition (Rank of a divisor is)

1. \( r(D) = -1 \) if \( |D| \) is empty.
2. \( r(D) \geq 0 \) if \( |D| \) is nonempty.
3. \( r(D) \geq k \) if \( |D - E| \) is nonempty for any effective \( E \) with \( \deg E = k \).

Remark

1. If \( X \) is smooth, then \( r(D) = \dim H^0(X, D) - 1 \).
2. If \( X \) is has multiple components, then \( r(D) \neq \dim H^0(X, D) - 1 \).

Remark

Ingredients of Stoll’s proof only use formal properties of \( r(D) \).
Formal ingredients of Stoll’s proof

Need:

(Clifford) \( r(K - D) \leq \frac{1}{2} \deg(K - D) \)

(Large rank) \( r(K - D) \geq g - r - 1 \)

(Recall, \( V \subset H^0(X_{\mathbb{Q}_p}, \Omega^1_X) \), \( \dim_{\mathbb{Q}_p} V \geq g - r \))
Semistable case

**Idea:** any section $s \in H^0(X, D)$ can be scaled to not vanish on a component (but may now have zeroes or poles at other components.)

**Divisors on graphs:**

![Divisor on a grid](image1.png)

![Divisor on a graph](image2.png)
**Semistable case**

**Idea:** any section \( s \in H^0(X, D) \) can be scaled to not vanish on a component (but may now have zeroes or poles at other components.)

**Divisors on graphs:**

![Diagram of divisors on graphs]

\[ \begin{array}{c}
\text{1} \\
\text{1} \\
\text{-2} \\
\text{0} \\
\text{1} \\
\end{array} \]

\[ \begin{array}{c}
\text{1} \\
\text{1} \\
\text{-2} \\
\text{0} \\
\text{1} \\
\end{array} \]
Semistable case

**Idea:** any section $s \in H^0(X, D)$ can be scaled to not vanish on a component (but may now have zeroes or poles at other components.)

**Divisors on graphs:**

![Graph 1](image1)

![Graph 2](image2)
Divisors on graphs

Definition (Rank of a divisor is)
1. \( r(D) = -1 \) if \(|D|\) is empty.
2. \( r(D) \geq 0 \) if \(|D|\) is nonempty
3. \( r(D) \geq k \) if \(|D - E|\) is nonempty for any effective \( E \) with \( \deg E = k \).

Remark
\( r(D) \geq 0 \)
Definition (Rank of a divisor is)

1. $r(D) = -1$ if $|D|$ is empty.
2. $r(D) \geq 0$ if $|D|$ is nonempty.
3. $r(D) \geq k$ if $|D - E|$ is nonempty for any effective $E$ with $\deg E = k$.

Remark

$r(D) \geq 1$
Semistable case – line bundles

Let $\mathcal{X}$ be a curve over $\mathbb{Z}_p$ with semistable special fiber $\mathcal{X}_{\mathbb{F}_p} = \bigcup X_i$.

**Definition (Divisor associated to a line bundle)**

Given $\mathcal{L} \in \text{Pic } \mathcal{X}$, define a divisor on $\Gamma$ by

$$
\sum_{\nu \in V(\Gamma)} (\deg \mathcal{L}_{X_i}) \nu_{X_i}.
$$
Let $\mathcal{X}$ be a curve over $\mathbb{Z}_p$ with semistable special fiber $X_{\mathbb{F}_p} = \bigcup X_i$.

**Definition (Divisor associated to a line bundle)**

Given $\mathcal{L} \in \text{Pic } \mathcal{X}$, define a divisor on $\Gamma$ by

$$
\sum_{v \in V(\Gamma)} (\deg \mathcal{L}_{X_i}) v_{X_i}.
$$

**Example:** $\mathcal{L} = \omega_{\mathcal{X}}$, $X_{\mathbb{F}_p}$ totally degenerate ($g(X_i) = 0$)
Semistable case – line bundles

Let $\mathcal{X}$ be a curve over $\mathbb{Z}_p$ with semistable special fiber $\mathcal{X}_{\mathbb{F}_p} = \bigcup X_i$.

**Definition (Divisor associated to a line bundle)**

Given $\mathcal{L} \in \text{Pic } \mathcal{X}$, define a divisor on $\Gamma$ by

$$\sum_{v \in V(\Gamma)} (\deg \mathcal{L}_{X_i}) v_{X_i}.$$  

**Example:** $\mathcal{L} = \mathcal{O}(H)$ (H a “horizontal” divisor on $\mathcal{X}$)
Semistable case – line bundles

Let $\mathcal{X}$ be a curve over $\mathbb{Z}_p$ with semistable special fiber $\mathcal{X}_{\mathbb{F}_p} = \bigcup X_i$.

**Definition (Divisor associated to a line bundle)**

Given $\mathcal{L} \in \text{Pic } \mathcal{X}$, define a divisor on $\Gamma$ by

$$\sum_{\nu \in V(\Gamma)} (\deg \mathcal{L}_{\nu_i}) \nu_{X_i}.$$ 

**Example:** $\mathcal{L} = \mathcal{O}(X_i), \quad X_i$
Divisors on graphs

Definition

For $\overline{D} \in \text{Div} \Gamma$, $r_{\text{num}}(\overline{D}) \geq k$ if $|\overline{D} - \overline{E}|$ is non-empty for every effective $\overline{E}$ of degree $k$.

Theorem (Baker, Norine)

- **Riemann-Roch** for $r_{\text{num}}$
- **Clifford’s theorem** for $r_{\text{num}}$
- **Specialization**: $r_{\text{num}}(\overline{D}) \geq r(D)$
- **Formal corollary**: $X(\mathbb{Q}) \leq \#X^{\text{sm}}(\mathbb{F}_p) + 2r$ (for $X$ totally degenerate).
Semistable case – main points

Remark (Main points)

1. Chip firing is the same as twising by $\mathcal{O}(X_i)$.
2. If $\exists s \in H^0(\mathcal{X}, \mathcal{L})$ and $\text{div } s = \sum H_i + \sum n_i X_i$, then

   $\mathcal{L} \otimes \mathcal{O}(-n_1 X_1) \otimes \cdots \otimes \mathcal{O}(-n_k X_k)$

   specializes to an effective divisor on $\Gamma$.
3. The firing sequence $(n_1, \ldots, n_n)$ wins the chip firing game.
Semistable but not totally degenerate – abelian rank

Problems when \( g(\Gamma) < g(\mathcal{X}) \). (E.g. rank can increase after reduction.)

**Definition (Abelian rank \( r_{ab} \))**

Let \( \mathcal{L} \in \mathcal{X} \) have specialization \( D \in \text{Div}\, \Gamma \). Then \( r_{ab}(\mathcal{L}) \geq k \) if

1. \(|D - E|\) is nonempty for any effective \( E \) with \( \text{deg} \, E = k \), and
2. for every \( \mathcal{L}_E \) specializing to \( E \), there exists some \((n_1, \ldots, n_k)\) such that

\[
\mathcal{L}' := \mathcal{L} \otimes \mathcal{L}_E^{-1} \otimes \mathcal{O}(n_1X_1) \otimes \cdots \otimes \mathcal{O}(n_kX_k)
\]

has effective specialization and such that \( H^0(X_i, \mathcal{L}'_{X_i}) \neq 0 \) for every component \( X_i \).
Main Theorem – abelian rank

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<th>Theorem (Katz-ZB)</th>
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<td><strong>Clifford’s theorem:</strong>  ( r_{ab}(K - D) \leq \frac{1}{2} \deg(K - D) )</td>
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<tr>
<td><strong>Specialization:</strong>  ( r_{ab}(K - D) \geq g - r. )</td>
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<td><strong>Formal corollary:</strong>  ( X(\mathbb{Q}) \leq # X^{\text{sm}}(\mathbb{F}_p) + 2r )  (for semistable curves.)</td>
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Final remarks

Remark
Also prove: **semistable case \(\Rightarrow\) general case.**

Remark (Néron models)
1. Suppose \(\mathcal{L} \in \text{Pic} \mathcal{X}\) and \(\deg (\mathcal{L}|_{\mathcal{X}_p}) = 0\).
2. \(r_{\text{num}}(\mathcal{L}) = 0\) if and only if \(\mathcal{L}|_{\mathcal{X}_p} \in \text{Pic}^0_{\mathcal{X}_p}\).
3. \(r_{\text{ab}}(\mathcal{L}) = 0\) if and only if the image of \(\mathcal{L}|_{\mathcal{X}_p}\) in \(\text{Pic}^0_{\widetilde{\mathcal{X}}_p}\) is the identity.

Remark (Toric rank)
1. Can also define \(r_{\text{tor}}\) – additionally require that sections agree at nodes
2. \(r_{\text{tor}}\) incorporates the toric part of Néron model