Rational points on curves and chip firing.

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Slides available at http://www.mathcs.emory.edu/~dzb/slides/

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Theorem (Faltings)

Let $X$ be a smooth curve over $\mathbb{Q}$ with genus at least 2. Then $X(\mathbb{Q})$ is finite.

Example

For $g \geq 2$, $y^2 = x^{2g+1} + 1$ has only finitely many solutions with $x, y \in \mathbb{Q}$. 

**Problem**

1. Given $X$, compute $X(\mathbb{Q})$ exactly.
2. Compute bounds on $\#X(\mathbb{Q})$.

**Conjecture (Uniformity)**

There exists a constant $N(g)$ such that every smooth curve of genus $g$ over $\mathbb{Q}$ has at most $N(g)$ rational points.

This would follow from standard conjectures (e.g. Lang’s conjecture, the higher dimensional analogue of Faltings’ theorem).
Coleman’s bound

**Theorem (Coleman)**

Let $X$ be a curve of genus $g$ and let $r = \text{rank}_\mathbb{Z} \text{Jac}_X(\mathbb{Q})$. Suppose $p > 2g$ is a prime of good reduction. Suppose $r < g$. Then

$$\#X(\mathbb{Q}) \leq \#X(\mathbb{F}_p) + 2g - 2.$$ 

**Remark**

1. A modified statement holds for $p \leq 2g$ or for $K \neq \mathbb{Q}$.
2. Note: this does not prove uniformity (since the first good $p$ might be large).
Stoll’s bound

Theorem (Stoll)

Let $X$ be a curve of genus $g$ and let $r = \text{rank}_{\mathbb{Z}} \text{Jac}_X(\mathbb{Q})$. Suppose $p > 2g$ is a prime of good reduction. Suppose $r < g$. Then

$$\#X(\mathbb{Q}) \leq \#X(\mathbb{F}_p) + 2r.$$
**Theorem (Lorenzini-Tucker, McCallum-Poonen)**

Let $X$ be a curve of genus $g$ and let $r = \text{rank}_\mathbb{Z} \text{Jac}_X(\mathbb{Q})$. Suppose $p > 2g$ is a prime. Suppose $r < g$.

Let $\mathcal{X}$ be a regular proper model of $X$. Then

$$\#X(\mathbb{Q}) \leq \#\mathcal{X}^{\text{sm}}(\mathbb{F}_p) + 2g - 2.$$ 

**Remark**

A recent improvement due to Stoll gives a uniform bound if $r \leq g - 3$ and $X$ is hyperelliptic.
Theorem (Katz-ZB)

Let $X$ be a curve of genus $g$ and let $r = \text{rank}_\mathbb{Z} \text{Jac}_X(\mathbb{Q})$. Suppose $p > 2g$ is a prime. Let $\mathcal{X}$ be a regular proper model of $X$. Suppose $r < g$. Then

$$\#X(\mathbb{Q}) \leq \#\mathcal{X}^\text{sm}(\mathbb{F}_p) + 2r.$$
Example (hyperelliptic curve with cuspidal reduction)

\[-2 \cdot 11 \cdot 19 \cdot 173 \cdot y^2 = (x - 50)(x - 9)(x - 3)(x + 13)(x^3 + 2x^2 + 3x + 4)\]

\[= x(x + 1)(x + 2)(x + 3)(x + 4)^3 \mod 5.\]

Analysis

1. \(X(\mathbb{Q})\) contains

\[\{\infty, (50, 0), (9, 0), (3, 0), (-13, 0), (25, 20247920), (25, -20247920)\}\]

2. \(\# X_5^{sm}(\mathbb{F}_5) = 5\)

3. \(7 \leq \# X(\mathbb{Q}) \leq \# X_5^{sm}(\mathbb{F}_5) + 2 \cdot 1 = 7\)

This determines \(X(\mathbb{Q})\).
Non-example

\[ y^2 = x^6 + 5 = x^6 \mod 5. \]

Analysis

1. \( X(\mathbb{Q}) \supset \{\infty^+, \infty^-\} \)
2. \( \mathcal{X}_{\text{sm}}(\mathbb{F}_5) = \{\infty^+, \infty^-, \pm(1, \pm 1), \pm(2, \pm 2^3), \pm(3, \pm 3^3), \pm(4, \pm 4^3)\} \)
3. \( 2 \leq \#X(\mathbb{Q}) \leq \#\mathcal{X}_{\text{sm}}(\mathbb{F}_5) + 2 \cdot 1 = 20 \)
Models

\[ y^2 = x^6 + 5 = x^6 \mod 5. \]

Note: no point can reduce to \((0, 0)\).
Models

\[ y^2 = x^6 + 5^2 \]

\[ = x^6 \mod 5 \]

Now: \((0, 5)\) reduces to \((0, 0)\). Local equation looks like \(xy = 5^2\)
$$y^2 = x^6 + 5^2$$

$$= x^6 \mod 5$$

Blow up. Local equation looks like $xy = 5$
Models

\[ y^2 = x^6 + 5^4 = x^6 \mod 5 \]

Blow up. Local equation looks like \( xy = 5^3 \)
Models

\[ y^2 = x^6 + 5^4 \]

\[ = x^6 \mod 5 \]

Blow up. Local equation looks like \( xy = 5 \)
Main Theorem

**Theorem (Katz-ZB)**

Let $X$ be a curve of genus $g$ and let $r = \text{rank}_{\mathbb{Z}} \text{Jac}_X(\mathbb{Q})$. Suppose $p > 2g$ is a prime. Let $\mathcal{X}$ be a regular proper model of $X$. Suppose $r < g$. Then

$$\#X(\mathbb{Q}) \leq \#\mathcal{X}^{\text{sm}}(\mathbb{F}_p) + 2r.$$
Chabauty’s method

\textbf{\((p\text{-adic integration})\) There exists} \( V \subset H^0(X_{\mathbb{Q}_p}, \Omega^1_X) \) \text{ with} 
\[ \dim_{\mathbb{Q}_p} V \geq g - r \] \text{ such that,}
\[ \int_P^Q \omega = 0 \quad \forall P, Q \in X(\mathbb{Q}), \omega \in V \]

\textbf{\textbf{(Coleman, via Newton Polygons}) Number of zeroes in a residue disc} \(D_P\) \text{ is} \(\leq 1 + n_P\), \text{ where} \(n_P = \# (\text{div} \omega \cap D_P)\)

\textbf{\textbf{(Riemann-Roch)}} \(\sum n_P = 2g - 2\).

\textbf{\textbf{(Coleman’s bound)}} \(\sum_{P \in X(\mathbb{F}_p)} (1 + n_P) = \# X(\mathbb{F}_p) + 2g - 2\).
Example (from McCallum-Poonen’s survey paper)

Example

\[ X : y^2 = x^6 + 8x^5 + 22x^4 + 22x^3 + 5x^2 + 6x + 1 \]

1. Points reducing to \( \tilde{Q} = (0, 1) \) are given by

\[ x = p \cdot t, \quad \text{where} \quad t \in \mathbb{Z}_p \]

\[ y = \sqrt{x^6 + 8x^5 + 22x^4 + 22x^3 + 5x^2 + 6x + 1} = 1 + x^2 + \cdots \]

2. \[
\int_{(0,1)}^{P_t} \frac{x \, dx}{y} = \int_0^t (x - x^3 + \cdots) \, dx
\]
Stoll’s idea: use multiple $\omega$

(Coleman, via Newton Polygons) Number of zeroes of $\int \omega$ in a residue class $D_p$ is $\leq 1 + n_p$, where $n_p = \# (\text{div} \omega \cap D_p)$

Let $\bar{n}_p = \min_{\omega \in V} \# (\text{div} \omega \cap D_p)$

(Example) $r \leq g - 2$, $\omega_1, \omega_2 \in V$

(Stoll’s bound) $\sum \bar{n}_p \leq 2r$. (Recall $\dim_{\mathbb{Q}_p} V \geq g - r$)
Stoll’s bound; proof.

Let $D = \sum \widehat{n}_P P$. Wanted: $\deg D \leq 2r$

(Clifford) If $H^0(X_{\mathbb{F}_p}, K - D') \neq 0$ then

$$\dim H^0(X_{\mathbb{F}_p}, D') \leq \frac{1}{2} \deg D' + 1$$

($D' = K - D$)

$$\frac{1}{2} \deg(K - D) + 1 \geq \dim H^0(X_{\mathbb{F}_p}, K - D)$$

(Assumption)

$$\dim H^0(X_{\mathbb{F}_p}, K - D) \geq g - r$$

(Recall $\dim_{\mathbb{Q}_p} V \geq g - r$)
Complications when $X_{\mathbb{F}_p}$ is singular

1. $\omega \in H^0(X, \Omega)$ may vanish along components of $X_{\mathbb{F}_p}$.
2. I.e. $H^0(X_{\mathbb{F}_p}, K - D) \neq 0 \not\Rightarrow D$ is special.
3. $\text{rank}(K - D) \neq \dim H^0(X_{\mathbb{F}_p}, K - D) - 1$

Summary

The relationship between $\dim H^0(X_{\mathbb{F}_p}, K - D)$ and $\deg D$ is less transparent and does not follow from geometric techniques.
Definition (Rank of a divisor is)

1. \( r(D) = -1 \) if \( |D| \) is empty.
2. \( r(D) \geq 0 \) if \( |D| \) is nonempty
3. \( r(D) \geq k \) if \( |D - E| \) is nonempty for any effective \( E \) with \( \deg E = k \).

Remark

1. If \( X \) is smooth, then \( r(D) = \dim H^0(X, D) - 1 \).
2. If \( X \) is has multiple components, then \( r(D) \neq \dim H^0(X, D) - 1 \).

Remark

Ingredients of Stoll’s proof only use formal properties of \( r(D) \).
Formal ingredients of Stoll’s proof

Need:

(Clifford) \[ r(K - D) \leq \frac{1}{2} \deg(K - D) \]

(Large rank) \[ r(K - D) \geq g - r - 1 \]

(Recall, \( V \subset H^0(X_{\mathbb{Q}_p}, \Omega_X^1), \dim_{\mathbb{Q}_p} V \geq g - r \))
Semistable case

**Idea:** any section \( s \in H^0(X, D) \) can be scaled to not vanish on a component (but may now have zeroes or poles at other components.)

**Divisors on graphs:**

![Graph 1](image1)

![Graph 2](image2)
Semistable case

**Idea:** any section \( s \in H^0(X, D) \) can be scaled to not vanish on a component (but may now have zeroes or poles at other components.)

**Divisors on graphs:**
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Divisors on graphs

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Remark
\( r(D) \geq 0 \)
Definition (Rank of a divisor is)

1. \( r(D) = -1 \) if \( |D| \) is empty.
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Remark

\( r(D) \geq 1 \)
Divisors on graphs

Definition

For $D \in \text{Div} \, \Gamma$, $r_{\text{num}}(D) \geq k$ if $|D - E|$ is non-empty for every effective $E$ of degree $k$.

Theorem (Baker, Norine)

- **Riemann-Roch** for $r_{\text{num}}$.
- **Clifford’s theorem** for $r_{\text{num}}$.
- **Specialization**: $r_{\text{num}}(\overline{D}) \geq r(D)$.
- **Formal corollary**: $X(\mathbb{Q}) \leq \#X^{\text{sm}}(\mathbb{F}_p) + 2r$ (for $X$ totally degenerate).
General case (not totally degenerate) – abelian rank

Problems when $g(\Gamma) < g(X)$. (E.g. rank can increase after reduction.)

**Definition (Abelian rank $r_{ab}$)**

After winning the chip firing game, we additionally require that the resulting divisor is equivalent to an effective divisor on that component.

**Theorem (Katz-ZB)**

- **Clifford’s theorem**: for $r_{ab}$
- **Specialization**: $r_{ab}(K - D) \geq g - r$.
- **Formal corollary**: $X(\mathbb{Q}) \leq \#X^{\text{sm}}(\mathbb{F}_p) + 2r$ (for semistable curves.)