Rational points on curves and chip firing.

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Slides available at http://www.mathcs.emory.edu/~dzb/slides/

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Theorem (Faltings)

Let \( C \) be a smooth curve over \( \mathbb{Q} \) with genus at least 2. Then \( C(\mathbb{Q}) \) is finite.

Example

For \( g \geq 2 \), \( y^2 = x^{2g+1} + 1 \) has only finitely many solutions with \( x, y \in \mathbb{Q} \).
Problem

1. Given $C$, compute $C(\mathbb{Q})$ exactly.
2. Compute bounds on $\#C(\mathbb{Q})$.

Conjecture (Uniformity)

There exists a constant $N(g)$ such that every smooth curve of genus $g$ over $\mathbb{Q}$ has at most $N(g)$ rational points.

This would follow from standard conjectures (e.g. Lang’s conjecture, the higher dimensional analogue of Faltings’ theorem).
Theorem (Coleman)

Let $X$ be a curve of genus $g$ and let $r = \text{rank}_{\mathbb{Z}} \text{Jac}_X(\mathbb{Q})$. Suppose $p > 2g$ is a prime of good reduction. Suppose $r < g$. Then

$$\#X(\mathbb{Q}) \leq \#X(\mathbb{F}_p) + 2g - 2.$$ 

Remark

1. A modified statement holds for $p \leq 2g$ or for $K \neq \mathbb{Q}$.
2. Note: this does not prove uniformity (since the first good $p$ might be large).
Stoll’s bound

**Theorem (Stoll)**

Let $X$ be a curve of genus $g$ and let $r = \text{rank}_\mathbb{Z} \text{Jac}_X(\mathbb{Q})$. Suppose $p > 2g$ is a prime of good reduction. Suppose $r < g$. Then

$$\#X(\mathbb{Q}) \leq \#X(\mathbb{F}_p) + 2r.$$
Theorem (Lorenzini-Tucker, McCallum-Poonen)

Let $X$ be a curve of genus $g$ and let $r = \text{rank}_\mathbb{Z} \text{Jac}_X(\mathbb{Q})$. Suppose $p > 2g$ is a prime. Suppose $r < g$.

Let $\mathcal{X}$ be a regular proper model of $C$. Then

$$\#X(\mathbb{Q}) \leq \#\mathcal{X}^{\text{sm}}(\mathbb{F}_p) + 2g - 2.$$ 

Remark

A recent improvement due to Stoll gives a uniform bound if $r \leq g - 3$. 
Main Theorem

Theorem (ZB-Katz)

Let \( X \) be a curve of genus \( g \) and let \( r = \text{rank}_\mathbb{Z} \text{Jac}_X(\mathbb{Q}) \). Suppose \( p > 2g \) is a prime. Let \( \mathcal{X} \) be a regular proper model of \( C \). Suppose \( r < g \). Then

\[
\#X(\mathbb{Q}) \leq \#\mathcal{X}^{\text{sm}}(\mathbb{F}_p) + 2r.
\]
Example (hyperelliptic curve with cuspidal reduction)

\[-2 \cdot 11 \cdot 19 \cdot 173 \cdot y^2 = (x - 50)(x - 9)(x - 3)(x + 13)(x^3 + 2x^2 + 3x + 4) = x(x + 1)(x + 2)(x + 3)(x + 4)^3 \mod 5.\]

Analysis

1. $X(\mathbb{Q})$ contains
   \[\{\infty, (50, 0), (9, 0), (3, 0), (-13, 0), (25, 20247920), (25, -20247920)\}\]

2. $\# X_{sm}^5(\mathbb{F}_5) = 5$

3. $7 \leq \# X(\mathbb{Q}) \leq \# X_{sm}^5(\mathbb{F}_5) + 2 \cdot 1 = 7$

This determines $X(\mathbb{Q})$
Non-example

\[ y^2 = x^6 + 5 \]
\[ = x^6 \mod 5. \]

Analysis

1. \[ X(\mathbb{Q}) \supset \{\infty^+, \infty^-\} \]
2. \[ \mathcal{X}^{\text{sm}}(\mathbb{F}_5) = \{\infty^+, \infty^-, \pm(1, \pm 1), \pm(2, \pm 2^3), \pm(3, \pm 3^3), \pm(4, \pm 4^3), \} \]
3. \[ 2 \leq \#X(\mathbb{Q}) \leq \#\mathcal{X}_5^{\text{sm}}(\mathbb{F}_5) + 2 \cdot 1 = 20 \]
Models

\[ y^2 = x^6 + 5 \]

\[ = x^6 \mod 5. \]

Note: no point can reduce to \((0, 0)\).
$y^2 = x^6 + 5^2$

$= x^6 \mod 5$

Now: $(0, 5)$ reduces to $(0, 0)$. Local equation looks like $xy = 5^2$
Models

\[ y^2 = x^6 + 5^2 \]

\[ = x^6 \mod 5 \]

Blow up. Local equation looks like \( xy = 5 \)
Models

\[ y^2 = x^6 + 5^4 \]

\[ = x^6 \mod 5 \]

Blow up. Local equation looks like \( xy = 5^3 \)
Models

\[ y^2 = x^6 + 5^4 \]
\[ = x^6 \mod 5 \]

Blow up. Local equation looks like \( xy = 5 \)
(\textbf{$p$-adic integration}) There exists $V \subset H^0(X_{\mathbb{Q}_p}, \Omega^1_X)$ with $\dim_{\mathbb{Q}_p} V \geq g - r$ such that,

$$\int_P^Q \omega = 0 \quad \forall P, Q \in X(\mathbb{Q}), \omega \in V$$

(\textbf{Coleman, via Newton Polygons}) Number of zeroes in a residue class $D_P$ is $\leq 1 + n_P$, where $n_P = \# (\text{div} \omega \cap D_P)$

(\textbf{Riemann-Roch}) $\sum n_P = 2g - 2$.

(\textbf{Coleman’s bound}) $\sum_{P \in X(\mathbb{F}_p)} (1 + n_P) = \# X(\mathbb{F}_p) + 2g - 2$. 
Example

\[ X : y^2 = x^6 + 8x^5 + 22x^4 + 22x^3 + 5x^2 + 6x + 1 \]

1. Points reducing to \( \tilde{Q} = (0, 1) \) are given by

\[
x = p \cdot t, \text{ where } t \in \mathbb{Z}_p
\]

\[
y = \sqrt{x^6 + 8x^5 + 22x^4 + 22x^3 + 5x^2 + 6x + 1} = 1 + x^2 + \cdots
\]

2. \[
\int_{(0,1)}^{P_t} \frac{xdx}{y} = \int_0^t (x - x^3 + \cdots)dx
\]
Stoll’s idea: use multiple $\omega$

(Coleman, via Newton Polygons) Number of zeroes of $\int \omega$ in a residue class $D_P$ is $\leq 1 + n_P$, where $n_P = \#(\text{div } \omega \cap D_P)$

Let $\tilde{n}_P = \min_{\omega \in V} \#(\text{div } \omega \cap D_P)$

(Example) $r \leq g - 2$, $\omega_1, \omega_2 \in V$

\[
\sum \tilde{n}_P \leq 2r. \quad (\text{Recall } \dim_{\mathbb{Q}_p} V \geq g - r)
\]
Let $D = \sum \widehat{n_P} P$. Wanted: $\deg D \leq 2r$

(Clifford) If $H^0(X_{\overline{K}_p}, K - D') \neq 0$ then

$$\dim H^0(X_{\overline{K}_p}, D') \leq \frac{1}{2} \deg D' + 1$$

($D' = K - D$)

$$\frac{1}{2} \deg(K - D) + 1 \geq \dim H^0(X_{\overline{K}_p}, K - D)$$

(Assumption)

$$\dim H^0(X_{\overline{K}_p}, K - D) \geq g - r$$

(Recall $\dim_{\overline{K}_p} V \geq g - r$)
Complications when $X_{\mathbb{F}_p}$ is singular

1. $\omega \in H^0(X, \Omega)$ may vanish along components of $X_{\mathbb{F}_p}$.
2. I.e. $H^0(X_{\mathbb{F}_p}, K - D) \neq 0 \nRightarrow D$ is special.
3. $\text{rank}(K - D) \neq \dim H^0(X_{\mathbb{F}_p}, K - D) - 1$

**Summary**

The relationship between $\dim H^0(X_{\mathbb{F}_p}, K - D)$ and $\deg D$ is less transparent and does not follow from geometric techniques.
Rank of a divisor

Definition (Rank of a divisor is)

1. \( r(D) = -1 \) if \( |D| \) is empty.
2. \( r(D) \geq 0 \) if \( |D| \) is nonempty
3. \( r(D) \geq k \) if \( |D - E| \) is nonempty for any effective \( E \) with \( \deg E = k \).

Remark

1. If \( X \) is smooth, then \( r(D) = \dim H^0(X, D) - 1 \).
2. If \( X \) is has multiple components, then \( r(D) \neq \dim H^0(X, D) - 1 \).

Remark

Ingredients of Stoll’s proof only use formal properties of \( r(D) \).
Formal ingredients of Stoll’s proof

Need:

(Clifford) \[ r(K - D) \leq \frac{1}{2} \deg(K - D) \]

(Large rank) \[ r(K - D) \geq g - r - 1 \]

(Recall, \( V \subset H^0(X_{\mathbb{Q}_p}, \Omega^1_X) \), \( \dim_{\mathbb{Q}_p} V \geq g - r \))
**Semistable case**

**Idea:** any section $s \in H^0(X, D)$ can be scaled to not vanish on a component (but may now have zeroes or poles at other components.)

**Divisors on graphs:**

![Diagram of a curve and its dual graph.](image)
Semistable case

**Idea:** any section \( s \in H^0(X, D) \) can be scaled to not vanish on a component (but may now have zeroes or poles at other components.)

**Divisors on graphs:**

\[ \begin{align*}
\text{Graph 1} & : \quad \bullet \bullet \bullet \\
\text{Graph 2} & : \quad \begin{array}{c}
1 \\
-2 \quad 0 \quad 2 \\
1 \\
1
\end{array}
\]
Semistable case

Idea: any section \( s \in H^0(X, D) \) can be scaled to not vanish on a component (but may now have zeroes or poles at other components.)

Divisors on graphs:
**Definition**

For $\overline{D} \in \text{Div } \Gamma$, $r_{\text{num}}(\overline{D}) \geq k$ if $|\overline{D} - \overline{E}|$ is non-empty for every effective $\overline{E}$ of degree $k$.

**Theorem (Baker, Norine)**

- **Riemann-Roch** for $r_{\text{num}}$.
- **Clifford's theorem** for $r_{\text{num}}$.
- **Specialization**: $r_{\text{num}}(\overline{D}) \geq r(D)$.
- **Formal corollary**: $X(\mathbb{Q}) \leq \#X^{\text{sm}}(\mathbb{F}_p) + 2r$ (for $X$ totally degenerate).
General case (not totally degenerate) – abelian rank

Problems when $g(\Gamma) < g(X)$. (E.g. rank can increase after reduction.)

**Definition (Abelian rank $r_{ab}$)**
After winning the chip firing game, we additionally require that the resulting divisor is equivalent to an effective divisor on that component.

**Theorem (Katz-ZB)**

- **Clifford’s theorem** holds for $r_{ab}$
- **Specialization**: $r_{ab}(K - D) \geq g - r$.
- **Formal corollary** $X(\mathbb{Q}) \leq \#X^{\text{sm}}(\mathbb{F}_p) + 2r$ (for semistable curves.)