Tropical geometry, $p$-adic integration, and uniformity.

David Zureick-Brown (Emory University)
Joe Rabinoff (Georgia Tech)
Eric Katz (Waterloo University)

Slides available at http://www.mathcs.emory.edu/~dzb/slides/

Special Session on Combinatorics and Algebraic Geometry

Fall Western Sectional Meeting
San Francisco State University

Oct 26, 2014
Theorem (Faltings, Vojta, Bombieri)

Let $X$ be a smooth curve over $\mathbb{Q}$ with genus at least 2. Then $X(\mathbb{Q})$ is finite.

Example

For $g \geq 2$, $y^2 = x^{2g+1} + 1$ has only finitely many solutions with $x, y \in \mathbb{Q}$. 
Uniformity

Problem

1. Given $X$, compute $X(\mathbb{Q})$ exactly.
2. Compute bounds on $\#X(\mathbb{Q})$.

Conjecture (Uniformity)

There exists a constant $N(g)$ such that every smooth curve of genus $g$ over $\mathbb{Q}$ has at most $N(g)$ rational points.

Theorem (Caporaso, Harris, Mazur)

Lang's conjecture $\Rightarrow$ uniformity.
Coleman’s bound

**Theorem (Coleman)**

Let $X$ be a curve of genus $g$ and let $r = \text{rank}_\mathbb{Z} \text{Jac}_X(\mathbb{Q})$. Suppose $p > 2g$ is a prime of **good reduction**. Suppose $r < g$. Then

$$\#X(\mathbb{Q}) \leq \#X(\mathbb{F}_p) + 2g - 2.$$ 

**Remark**

1. A modified statement holds for $p \leq 2g$ or for $K \neq \mathbb{Q}$.
2. Note: this does not prove uniformity (since the first good $p$ might be large).

**Tools**

$p$-adic integration and Riemann–Roch
Stoll’s bound

Theorem (Stoll)

Let $X$ be a curve of genus $g$ and let $r = \text{rank}_\mathbb{Z} \text{Jac}_X(\mathbb{Q})$. Suppose $p > 2g$ is a prime of good reduction. Suppose $r < g$. Then

$$\#X(\mathbb{Q}) \leq \#X(\mathbb{F}_p) + 2r.$$ 

Tools

$p$-adic integration, Riemann–Roch, and Clifford’s theorem
Bad reduction bound

Theorem (Lorenzini-Tucker, McCallum-Poonen)

Let $X$ be a curve of genus $g$ and let $r = \text{rank}_\mathbb{Z} \text{Jac}_X(\mathbb{Q})$. Suppose $p > 2g$ is a prime. Suppose $r < g$.

Let $\mathcal{X}$ be a regular proper model of $X$. Then

$$\#X(\mathbb{Q}) \leq \#\mathcal{X}^{\text{sm}}(\mathbb{F}_p) + 2g - 2.$$ 

Remark (Still doesn’t prove uniformity)

$\#\mathcal{X}^{\text{sm}}(\mathbb{F}_p)$ can contain an $n$-gon, for $n$ arbitrarily large.

Tools

$p$-adic integration and arithmetic Riemann–Roch ($\mathcal{K} \cdot \mathcal{X}_p = 2g - 2$)
Theorem (Katz-ZB)

Let $X$ be a curve of genus $g$ and let $r = \text{rank}_\mathbb{Z} \text{Jac}_X(\mathbb{Q})$. Suppose $p > 2g$ is a prime. Let $\mathcal{X}$ be a regular proper model of $X$. Suppose $r < g$. Then

$$\#X(\mathbb{Q}) \leq \#\mathcal{X}^{\text{sm}}(\mathbb{F}_p) + 2r.$$ 

Remark

Still doesn't prove uniformity.

Tools

$p$-adic integration and Clifford's theorem for graphs
Stoll’s hyperelliptic uniformity theorem

**Theorem (Stoll)**

Let $X$ be a hyperelliptic curve of genus $g$ and let $r = \text{rank}_{\mathbb{Z}} \text{Jac}_X(\mathbb{Q})$. Suppose $r < g - 2$.

Let $\mathcal{X}$ be a stable proper model of $X$. Then

$$\#X(\mathbb{Q}) \leq 8(r + 4)(g - 1) + \max\{1, 4r\} \cdot g$$

**Tools**

- $p$-adic integration on annuli
- Comparison of different analytic continuations of $p$-adic integration
Main Theorem (partial uniformity for non-hyperelliptic curves)

**Theorem (Katz, Rabinoff, ZB)**

Let $X$ be any curve of genus $g$ and let $r = \text{rank}_{\mathbb{Z}} \text{Jac}_X(\mathbb{Q})$. Suppose $r \leq g - 2$. Let $d = 3(g+1)^2$ and let $p \geq 2g + d$. Then

$$\#X(\mathbb{Q}) \leq 2gp^{d/2} + (2g - 2)(p^2 + 2) + 2 \cdot g^g(6g - 6)(4g - 4).$$

**Tools**

- $p$-adic integration on annuli
- Comparison of different analytic continuations of $p$-adic integration
- Rabinoff’s bounds for Laurent series
- Tropical canonical bundle
Corollary ((Partially) effective Manin-Mumford)

There is an effective constant $N(g)$ such that if $g(X) = g$, then

$$\# (X \cap \mathrm{Jac}_{X,\text{tors}})(\mathbb{Q}) \leq N(g)$$

Corollary (In progress)

There is an effective constant $N'(g)$ such that if $g(X) = g > 3$ and $X$ has totally degenerate, trivalent reduction mod 2, then

$$\# (X \cap \mathrm{Jac}_{X,\text{tors}})(\mathbb{C}) \leq N'(g)$$
Models – semistable example

\[ y^2 = (x(x - 1)(x - 2))^3 - 5 \]

\[ = (x(x - 1)(x - 2))^3 \mod 5. \]

Note: no point can reduce to \((0, 0)\). Local equation looks like \(xy = 5\).
Models – semistable example (not regular)

\[ y^2 = (x(x - 1)(x - 2))^3 - 5^4 \]

\[ = (x(x - 1)(x - 2))^3 \mod 5 \]

Now: \((0, 5^2)\) reduces to \((0, 0)\). Local equation looks like \(xy = 5^4\)
$$y^2 = (x(x - 1)(x - 2))^3 - 5^4$$

$$= (x(x - 1)(x - 2))^3 \pmod{5}$$

Blow up. Local equation looks like $xy = 5^3$
Models – semistable example (regular at (0,0))

\[ y^2 = (x(x - 1)(x - 2))^3 - 5^4 \]

\[ = (x(x - 1)(x - 2))^3 \mod 5 \]

Blow up. Local equation looks like \(xy = 5\)
(\textit{p-adic integration}) There exists \( V \subset H^0(X_{\mathbb{Q}_p}, \Omega^1_X) \) with \( \dim_{\mathbb{Q}_p} V \geq g - r \) such that,

\[
\int_P^Q \omega = 0 \quad \forall P, Q \in X(\mathbb{Q}), \omega \in V
\]

(\textit{Coleman, via Newton Polygons}) Number of zeroes in a residue disc \( D_P \) is \( \leq 1 + n_P \), where \( n_P = \# (\text{div} \omega \cap D_P) \)

(\textit{Riemann-Roch}) \( \sum n_P = 2g - 2 \).

(\textit{Coleman’s bound}) \( \sum_{P \in X(\mathbb{F}_p)} (1 + n_P) = \# X(\mathbb{F}_p) + 2g - 2 \).
Example (from McCallum-Poonen’s survey paper)

\[ X : y^2 = x^6 + 8x^5 + 22x^4 + 22x^3 + 5x^2 + 6x + 1 \]

1. Points reducing to \( \tilde{Q} = (0, 1) \) are given by

\[
\begin{align*}
    x &= p \cdot t, \text{ where } t \in \mathbb{Z}_p \\
    y &= \sqrt{x^6 + 8x^5 + 22x^4 + 22x^3 + 5x^2 + 6x + 1} = 1 + x^2 + \cdots
\end{align*}
\]

2. \[
\int_{(0,1)}^{P_t} \frac{x}{y} = \int_0^t (x - x^3 + \cdots)dx
\]
(Chabauty’s method) There exists $V \subset H^0(X_{\mathbb{Q}_p}, \Omega^1_X)$ with $\dim_{\mathbb{Q}_p} V \geq g - r$ such that,

$$\int_P^Q \omega = 0 \quad \forall P, Q \in X(\mathbb{Q}), \omega \in V$$

(Coleman, via Newton Polygons) Number of zeroes in a residue disc $D_P$ is $\leq 1 + n_P$, where $n_P = \#(\text{div} \omega \cap D_P)$

(Riemann-Roch) $\sum n_P = 2g - 2$.

(Coleman’s bound) $\sum_{P \in X(\mathbb{F}_p)} (1 + n_P) = \#X(\mathbb{F}_p) + 2g - 2$. 

(p-adic integration)
Analytic continuation of integrals

(Residue Discs.)
\[ P \in \mathcal{X}^\text{sm}(\mathbb{F}_p), \ t : D_P \cong p\mathbb{Z}_p, \ \omega|_{D_P} = f(t)dt \]

(Integrals on a disc.)
\[ Q, R \in D_P, \quad \int_{Q}^{R} \omega := \int_{t(Q)}^{t(R)} f(t)dt. \]

(Integrals between discs.)
\[ Q \in D_{P_1}, \ R \in D_{P_2}, \quad \int_{Q}^{R} \omega := ? \]
Analytic continuation of integrals via Abelian varieties

(Integrals between discs.)

\[ Q \in D_{P_1}, \ R \in D_{P_2}, \ \int_Q^R \omega := ? \]

(Albanese map.)

\[ \iota : X \hookrightarrow \text{Jac} X, \ Q \mapsto [Q - \infty] \]

(Abelian integrals via functorality and additivity.)

\[ \int_Q^R \iota^* \omega = \int_{\iota(Q)}^{\iota(R)} \omega = \int_{[Q-\infty]}^{[R-\infty]} \omega = \int_0^{[R-Q]} \omega = \frac{1}{n} \int_0^{n[R-Q]} \omega \]
Analytic continuation of integrals via Frobenius

(Integrals between discs.)

\[ Q \in D_{P_1}, \ R \in D_{P_2}, \int_Q^R \omega := ? \]

(Abelian integrals via functorality and Frobenius.)

\[ \int_Q^R \omega = \int_Q^{\phi(Q)} \omega + \int_{\phi(Q)}^{\phi(R)} \omega + \int_{\phi(R)}^R \omega \]

(Very clever trick (Coleman))

\[ \int_{\phi(Q)}^{\phi(R)} \omega_i = \int_Q^R \phi^* \omega = \sum_j \int_Q^R a_{ij} \omega_j \]
Comparison of integrals

**Facts**

1. For $X$ with good reduction, the **Abelian** and **Coleman** integrals agree.
2. A mystery. The associated Berkovich curve is contractable.
3. For $X$ with bad reduction they differ.

**Theorem (Stoll)**

There exist linear functions $a(\omega), c(\omega)$ such that

\[
\oint_R \omega - \oint_Q \omega = a(\omega) (\log(t(R)) - \log(t(Q))) + c(\omega) (t(Q) - t(R))
\]
Assumption

Assume $\mathcal{X}/\mathbb{Z}_p$ is stable, but not regular.

(Residue Discs.)

$$P \in \mathcal{X}^{\text{sm}}(\overline{\mathbb{F}_p}), \ t: D_P \cong p\mathbb{Z}_p, \ \omega|_{D_P} = f(t)dt$$

(Residue Annuli.)

$$P \in \mathcal{X}^{\text{sing}}(\overline{\mathbb{F}_p}), \ t: D_P \cong p\mathbb{Z}_p - p^r\mathbb{Z}_p, \omega|_{D_P} = f(t, t^{-1})dt$$

(Integrals on an annulus are multivalued.)

$$\int_Q^R \omega := \int_{t(Q)}^{t(R)} f(t, t^{-1})dt = \cdots + a(\omega) \log t + \cdots$$

(Cover the annulus with discs)

Each analytic continuation implicitly chooses a branch of log.
(Abelian integrals.) Analytically continue via Albanese.
\[ \oint_{R}^{Q} \omega := \int_{t(R)}^{t(Q)} f(t, t^{-1}) dt = \cdots + a(\omega) \log_{ab} t + \cdots \]

(Berkovich-Coleman integrals.) Analytically continue via Frobenius.
\[ \int_{R}^{Q} \omega := \int_{t(R)}^{t(Q)} f(t, t^{-1}) dt = \cdots + a(\omega) \log_{\text{Col}} t + \cdots \]

(Stoll’s theorem.)
\[ \oint_{Q}^{R} \omega - \int_{Q}^{R} \omega = a(\omega) (\log_{ab}(r(R)) - \log_{ab}(t(Q))) + c(\omega) (t(Q) - t(R)) \]
Theorem (Katz, Rabinoff, ZB)

The difference \( \log_{Col} - \log_{ab} \) is the unique homomorphism that takes the value

\[
\int_{\gamma} \omega
\]

on \( \text{Trop}(\gamma) \), where \( \text{Trop}: G(\mathbb{K}) \to T(\mathbb{K})/T(\mathcal{O}) \).

\[
\begin{array}{ccc}
T & \downarrow & \text{Jac}_X \\
\Lambda & \rightarrow & G \\
& \downarrow & \\
& B & \\
\end{array}
\]

\( T = \text{torus} \), \( \Lambda = \text{discrete} \), and \( B = \text{Abelian w/ good reduction} \).
Proof of uniformity

(Ignore the log and comparison terms (Stoll’s idea))

\[ r < g - 2 \] and linear algebra allows one to find \( \omega \) with no log term, and same abelian and Coleman integrals.

(Bounds on Annuli, via Rabinoff)

\[ \int \omega \] is a Laurent series

\# zeroes is bounded by the number of zeroes \textit{and} poles of \( \omega \)

(Global step)

- \( \omega \) gives a section of the tropical canonical bundle on the dual graph.
- The order of the pole of \( \omega \) at a node is the slope of the section of the tropical canonical bundle on the corresponding edge of the dual graph.