MATH 250 HANDOUT 6 - INDUCTION

(1) Prove that for any integer \( n \geq 1 \),
\[ 2^0 + 2^1 + \ldots + 2^{n-1} = 2^n - 1. \]

(2) Prove that for any integer \( n \geq 1 \), \( n^2 \) is the sum of the first \( n \) odd integers. (For example, \( 3^2 = 1 + 3 + 5 \).)

(3) Show that \( 7^n - 1 \) is divisible by 6 for all integers \( n \geq 0 \).

(4) Prove that \( n^5 - 5n^3 + 4n \) is divisible by 120 for all integers \( n \geq 1 \).

(5) Prove that
\[ n^9 - 6n^7 + 9n^5 - 4n^3 \]
is divisible by 8640 for all integers \( n \geq 1 \).

(6) Prove that
\[ n^2 \mid ((n + 1)^n - 1) \]
for all integers \( n \geq 1 \).

(7) Show that
\[ (x - y) \mid (x^n - y^n) \]
for all integers \( n \geq 1 \).

(8) Use the result of the previous problem to show that for all integers \( n \geq 1 \)
\[ 8767^{2345} - 8101^{2345} \]
is divisible by 666.

(9) Show that
\[ 2903^n - 803^n - 464^n + 261^n \]
is divisible by 1897 for all integers \( n \geq 1 \).

(10) Prove that if \( n \) is an even natural number, then the number \( 13^n + 6 \) is divisible by 7.

(11) Prove that for every \( n \in \mathbb{Z}_{\geq 2} \), \( n^3 - n \) is a multiple of 6.

(12) Prove that \( n! \geq 3^n \) for all integers \( n \geq 7 \).

(13) Prove that \( 2^n \geq n^2 \) for all integers \( n \geq 4 \).

(14) Consider the sequence defined by \( a_1 = 1 \) and \( a_n = \sqrt{2a_{n-1}} \). Prove that \( a_n < 2 \) for all integers \( n \geq 1 \).

(15) Prove that the equation \( x^2 + y^2 = z^n \) has a solution in positive integers \( x, y, z \) for all integers \( n \geq 1 \).

(16) Prove that \( n^3 + (n + 1)^3 + (n + 2)^3 \) is divisible by 3 for all integers \( n \geq 1 \).

(17) Prove that
\[ \frac{1}{n + 1} + \frac{1}{n + 2} + \ldots + \frac{1}{3n + 1} > 1 \]
for all integers \( n \geq 1 \).

(18) Prove that
\[ \frac{4^n}{n + 1} \leq \frac{(2n)!}{(n!)^2} \]
for all integers \( n \geq 1 \).
(19) Consider the Fibonacci sequence \( \{F_n\} \) defined by \( F_0 = 0, F_1 = 1, F_{n+1} = F_n + F_{n-1}, n \geq 1 \). Prove that each of the following statements is true for all integers \( n \geq 1 \).

(a) \( F_1 + F_3 + F_5 + \ldots + F_{2n-1} = F_{2n} \)

(b) \( F_2 + F_4 + F_6 + \ldots + F_{2n} = F_{2n+1} - 1 \)

(c) \( F_n < 2^n \)

(d) \( F_{n-1}F_{n+1} = F_n^2 + (-1)^{n+1} \).

(e) Let \( \alpha = \frac{1 + \sqrt{5}}{2} \) and \( \beta = \frac{1 - \sqrt{5}}{2} \). Prove that \( F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}} \). (Hint: first prove by, for example, direct calculation that \( \alpha \) and \( \beta \) are solutions of the equation \( x^2 - x - 1 = 0 \).)

(f) Prove that \( F_1^2 + \cdots + F_n^2 = F_n F_{n+1} \).

(g) Find a formula for \( F_1 + \cdots + F_n \) and prove it via induction.

(20) Prove that \( n! > 2^n \) for all integers \( n \geq 4 \).

(21) Prove that the expression \( 3^{3n+3} - 26n - 27 \) is a multiple of 169 for all positive integers \( n \).

(22) Prove that if \( k \) is odd, then \( 2^{n+2} \) divides \( k^{2^n} - 1 \) for all positive integers \( n \).

(23) Prove that

\[
\frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}
\]

is an integer for all integers \( n \geq 0 \).
(24) Let \(a_n\) be the sequence defined by \(a_1 := 1, a_n := na_{n-1}\). Prove that \(a_n = n!\).

(25) Let \(a_n\) be the sequence defined by \(a_1 := 2, a_n := 2a_{n-1}\). Prove that \(a_n = 2^n\).

(26) Prove that \(3^n\) is odd for every non-negative integer.

(27) Prove that \(n(n-1)\) is even for every positive integer \(n\).

(28) Prove that \(n^3 + 2n\) is a multiple of 3 for every positive integer \(n\).

(29) \[\frac{1}{1^2} + \frac{1}{2^3} + \cdots + \frac{1}{(n-1)n} = \frac{n-1}{n}\]

(30) Prove that \(1^2 + 4^2 + 7^2 + \cdots + (3n-2)^2 = \frac{1}{2}n(6n^2 - 3n - 1)\) for \(n \in \mathbb{Z}_{\geq 1}\).

(31) Prove that \(2^2 + 5^2 + 8^2 + \cdots + (3n-1)^2 = \frac{1}{2}n(6n^2 + 3n - 1)\) for \(n \in \mathbb{Z}_{\geq 1}\).

(32) Prove that \(1^3 + 2^3 + 3^3 + \cdots + n^3 = (1+2+3+\cdots+n)^2\) (Hint: use \(1+2+3+\cdots+k = \frac{k(k+1)}{2}\)).

(33) Let \(n \in \mathbb{Z}_{\geq 0}\). Prove that \(\sum_{i=1}^{n} i = \frac{n(n+1)}{2}\).

(34) Let \(n \in \mathbb{Z}_{\geq 0}\). Prove that \(\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}\).

(35) Let \(n \in \mathbb{Z}_{\geq 0}\). Prove that \(\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}\).

(36) Let \(n \in \mathbb{Z}_{\geq 0}\). Find a formula for \(\sum_{i=1}^{n} i^4\). Prove that your formula is correct.

(37) Prove that \(2^n > n^2\) for \(n \geq 5\) for \(n \in \mathbb{Z}_{\geq 1}\).

In the following problems, let \(\binom{n}{k} = \frac{n!}{(n-k)!k!}\).

(38) Let \(x\) and \(y\) be variables. Prove that
\[
(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.
\]

(39) Prove that
\[
\binom{n}{k} = \binom{n-1}{k} - \binom{n-1}{k-1}.
\]