MATH 250 HANDOUT 15 - COMPOSITIONS AND INJECTIVITY/SURJECTIVITY

(1) Let \( f: \mathbb{R} \to \mathbb{R} \) be the function \( f(x) = \frac{1}{1+x^2} \) and let \( g: \mathbb{R} \to \mathbb{R} \) be the function \( g(x) = e^x \).

(a) What is \( g \circ f(0) \)?
(b) What is \( f \circ g(0) \)?
(c) Give a formula for \( f \circ g \) and \( g \circ f \).

(2) Let \( f: \mathbb{R} \to \mathbb{Z} \) be the function \( f(x) = \lfloor x \rfloor \) (i.e., round \( x \) down to the nearest integer) and let \( g: \mathbb{Z} \to \mathbb{Z} \) be the function \( g(n) = \) ‘the number of distinct prime factors of \( n \)’.
(So \( g(0) = g(1) = 0, g(4) = 1, g(6) = 2 \))

(a) What is \( g \circ f(\pi) \)?
(b) What is \( g \circ f(91.1023124) \)?
(c) Is \( g \circ f \) injective? Surjective?

(3) Let \( f: \mathbb{Z} \to P(\mathbb{Z}) \) be the function \( f(n) = n \) and let \( g: P(\mathbb{Z}) \to P(\mathbb{Z}) \) be the function \( g(S) = S \cap \{1\} \).

(a) What is \( g \circ f(0) \)?
(b) What is \( g \circ f(1) \)?
(c) Give a formula for \( g \circ f \).
(4) Let $f: A \to B$ and $g: B \to C$ be functions. Prove or disprove each of the following:

(a) If $f$ and $g$ are injections, then $gf$ is an injection.
(b) If $f$ and $g$ are surjections, then $gf$ is a surjection.
(c) If $f$ and $g$ are bijections, then $gf$ is a bijection.
(d) If $gf$ is an injection, then $f$ and $g$ are injections.
(e) If $gf$ is a surjection, then $f$ and $g$ are surjections.
(f) If $gf$ is a bijection, then $f$ and $g$ are bijections.
(g) If $gf$ is an injection, then $f$ is an injection.
(h) If $gf$ is an injection, then $g$ is an injection.
(i) If $gf$ is a surjection, then $f$ is a surjection.
(j) If $gf$ is a surjection, then $g$ is a surjection.
(k) If $gf$ is a bijection, then $f$ is a bijection.
(l) If $gf$ is a bijection, then $g$ is a bijection.
(m) If $gf$ is an injection and $g$ is invertible, then $f$ is an injection.