integers less than $10^{10}$. So although just about everyone believes that it is true, no general proof seems forthcoming in the immediate future.

**Exercises 5.4**

1. Prove Lemma 5.4.2.

2. Let $n \in \mathbb{Z}, n > 1$. Prove that if $n$ is not divisible by any prime number less than or equal to $\sqrt{n}$, then $n$ is a prime number.

3. Let $n$ be a positive integer greater than 1 with the property that whenever $n$ divides a product $ab$ where $a, b \in \mathbb{Z}$, then $n$ divides $a$ or $n$ divides $b$. Prove that $n$ is a prime number.

4. Prove Corollary 5.4.5.

5. Prove Corollary 5.4.6.

6. (a) Prove that $\sqrt{3}$ is irrational.
   (b) Prove that $\sqrt{2}$ is irrational.
   (c) Prove that $\sqrt[3]{2}$ is irrational for every $n \in \mathbb{Z}, n \geq 2$.
   (d) Prove that if $p$ is a prime number, then $\sqrt[p]{p}$ is irrational for every $n \in \mathbb{Z}, n \geq 2$.
   (e) Let $n, m \in \mathbb{Z}, n \geq 2$. Prove that if $m$ is not the $n$th power of an integer, then $\sqrt[n]{m}$ is irrational.

7. (a) Prove that $\log_{10} 3$ is irrational.
   (b) Prove that if $r$ is a rational number such that $r > 1$ and $r \neq 10^n$ for any positive integer $n$, then $\log_{10} r$ is irrational.

8. Write the following integers in standard form:
   (a) 594  
   (b) 1,400  
   (c) 42,750  
   (d) 191,737

9. Let $n \in \mathbb{Z}, n \geq 1$. Prove that $n$ is a perfect square if and only if, when $n$ is written in standard form, all of the exponents are even.

10. Let $a, b \in \mathbb{Z}$.
    (a) Prove that if $a^2 \mid b^2$, then $a \mid b$.
    (b) Prove that if $a^n \mid b^n$ for some positive integer $n$, then $a \mid b$.

11. (a) Let $a, b \in \mathbb{Z}$ such that $(a, b) = 1$. Suppose that $ab = x^2$ for some $x$ in $\mathbb{Z}$. Prove that $a = y^2$ and $b = z^2$ for some $y$ and $z$ in $\mathbb{Z}$.
    (b) Show that part (a) is false without the assumption that $a$ and $b$ are relatively prime.
    (c) Let $a, b \in \mathbb{Z}$ such that $(a, b) = 1$. Suppose that $ab = x^n$ for some $x$ in $\mathbb{Z}$ and some positive integer $n$. Prove that $a = y^n$ and $b = z^n$ for some $y$ and $z$ in $\mathbb{Z}$.
    (d) Let $a, b, c \in \mathbb{Z}^+$ such that $(a, b) = (a, c) = (b, c) = 1$. Suppose that $abc = x^2$ for some $x$ in $\mathbb{Z}^+$. Prove that $a, b, c$ are all squares in $\mathbb{Z}^+$. 
(e) Let \(a_1, a_2, \ldots, a_n \in \mathbb{Z}^+\) such that \((a_i, a_j) = 1\) if \(i \neq j\). Suppose that
\[ a_1 a_2 \ldots a_n = x^2 \]
for some \(x \in \mathbb{Z}^+\). Prove that each \(a_i\) is a square in \(\mathbb{Z}^+\).

12. Prove that if a positive integer of the form \(2^n + 1\) is prime, then \(m\) is a power of 2.

13. Prove that 2 is the only prime of the form \(n^3 + 1\).

14. Prove that if \(2^n - 1\) is prime, then \(n\) is prime.

15. Investigate the following statement:

If \(n\) is any positive integer, then \(n^2 + n + 41\) is always a prime number.

If you think it is true, give a proof; if false, give a counterexample.

16. Let \(a, b \in \mathbb{Z}^+\), \(a > 1, b > 1\). Let \(a = p_1^{e_1} \cdot p_2^{e_2} \cdot \ldots \cdot p_r^{e_r}\) and
\(b = p_1^{f_1} \cdot p_2^{f_2} \cdot \ldots \cdot p_r^{f_r}\), where \(p_1, p_2, \ldots, p_r\) are primes and \(m\) and \(n\) are nonnegative integers, for \(i = 1, 2, \ldots, r\). Let \(l_i = \min(m, n)\). Prove that
\[(a, b) = p_1^{l_1} \cdot p_2^{l_2} \cdot \ldots \cdot p_r^{l_r}.

17. Use Exercises 8 and 16 to find the greatest common divisor of 1,400 and 42,750.

18. Prove that if \(a\) is a positive integer of the form \(4n + 3\), then at least one prime divisor of \(a\) is of the form \(4n + 3\).

19. Prove that if \(a\) is a positive integer of the form \(3n + 2\), then at least one prime divisor of \(a\) is of the form \(3n + 2\).

20. Prove that there are infinitely many primes of the form \(3n + 2\), \(n \in \mathbb{Z}^+\).

21. Prove that there are infinitely many primes of the form \(6n + 5\), \(n \in \mathbb{Z}^+\).

22. Let \(n\) be a positive integer. Prove that the binomial coefficient \(\binom{2n}{n}\) is divisible by every prime \(p\) such that \(n < p \leq 2n\) but is not divisible by \(p^2\).

23. Let \(p\) be a prime number and \(t\) a positive integer. Let \(a \in \mathbb{Z}\). Suppose that
\(a\) divides \(p^t\). Prove that \(a = p^k\) for some \(k \in \mathbb{Z}\), \(1 \leq k \leq t\).

24. Let \(n, m \in \mathbb{Z}\), \((n, m) = 1\). Suppose that \(d\) is a positive divisor of \(nm\).
Prove that there exist positive integers \(d_1\) and \(d_2\) such that \(d = d_1 d_2\) where
\(d_1\) divides \(n\) and \(d_2\) divides \(m\).

25. If \(n\) is a positive integer, let \(\tau(n)\) denote the number of positive divisors of \(n\). So, for example, \(\tau(1) = 1, \tau(2) = 2, \tau(3) = 2, \tau(4) = 3, \tau(5) = 2, \tau(6) = 4\).

(a) Prove that if \(p\) is a prime number and \(t\) is a positive integer, then
\(\tau(p^t) = t + 1\).

(b) Let \(n, m \in \mathbb{Z}\), \((n, m) = 1\). Prove that \(\tau(nm) = \tau(n) \tau(m)\).

(c) Let \(n \in \mathbb{Z}\). Let \(n = p_1^{\gamma_1} p_2^{\gamma_2} \ldots p_r^{\gamma_r}\) be the prime factorization of \(n\).
Prove that \(\tau(n) = (\gamma_1 + 1)(\gamma_2 + 1) \ldots (\gamma_r + 1)\).