\[ s(2k) = (-1)^{k+1} \frac{(2\pi)^{2k}}{2(2k)!} B_{2k}. \]

As an example, we see that the series \( \sum_{n=1}^{\infty} \frac{1}{n^4} \) converges to \( \pi^4/90 \).

**Exercises 5.2**

1. Prove the following formulas using mathematical induction.
   (a) \( 1 + 3 + 5 + \ldots + (2n - 1) = n^2. \)
   (b) \( 1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n + 1)(2n + 1)}{6}. \)
   (c) \( 1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{n^2(n + 1)^2}{4}. \)

2. Prove the following:
   (a) \( 1^2 + 3^2 + 5^2 + \ldots + (2n - 1)^2 = \frac{(2n - 1)(2n)(2n + 1)}{6}. \)
   (b) \( 2^2 + 4^2 + 6^2 + \ldots + (2n)^2 = \frac{(2n)(2n + 1)(2n + 2)}{6}. \)

3. Prove that if \( a \) is any real number except 1, then
   \[ 1 + a + a^2 + a^3 + \ldots + a^n = \frac{(a^{n+1} - 1)}{a - 1}. \]

4. (a) Prove that \( 2^n > n^2 \) for all integers \( n \geq 5 \).
    (b) Prove that \( 2^n < n! \) for all \( n \geq 4 \).

5. Let \( a, b_1, b_2, \ldots, b_n \in \mathbb{Z} \). Prove that \( a(b_1 + b_2 + \ldots + b_n) = ab_1 + ab_2 + \ldots + ab_n \).

6. Let \( f: \mathbb{Z}^+ \to \mathbb{Z}^+ \) be defined recursively by \( f(1) = 1 \) and \( f(n + 1) = f(n) + 2^n \) for all \( n \in \mathbb{Z}^+ \). Prove that \( f(n) = 2^n - 1 \).

7. Let \( f: \mathbb{Z}^+ \to \mathbb{R} \) be defined recursively by \( f(1) = 1 \) and \( f(n + 1) = \sqrt{2 + f(n)} \) for all \( n \in \mathbb{Z}^+ \). Prove that \( f(n) < 2 \) for all \( n \in \mathbb{Z}^+ \).

8. The **Fibonacci numbers** \( f_n, n = 1, 2, 3, \ldots, \), are defined recursively by the formulas \( f_1 = 1, f_2 = 1, f_n = f_{n-1} + f_{n-2} \) for \( n \geq 3 \).
   (a) Write out the first ten Fibonacci numbers.
   (b) Compute \( f_1 + f_2, f_1 + f_2 + f_3, f_1 + f_2 + f_3 + f_4 \).
   (c) Derive a formula for the sum of the first \( n \) Fibonacci numbers and prove it by induction.
   (d) Prove that \( f_1^2 + f_2^2 + \ldots + f_n^2 = f_nf_{n+1} \) for all \( n \geq 1 \).

9. Let \( H_n \) be the number of handshakes required if in a group of \( n \) people each person shakes with every other person exactly once.
   (a) Compute \( H_0, \ldots, H_5 \).
(c) Find an explicit formula for \( H_n \).

10. Let \( A_1, A_2, \ldots, A_n \) be a collection of finite mutually disjoint sets. Prove that

\[
\left| \bigcup_{i=1}^{n} A_i \right| = \sum_{i=1}^{n} |A_i|.
\]

(This is Corollary 2.3.5 of Section 2.3. You may assume Theorem 2.3.4 given in that section.)

11. Let \( * \) be an associative binary operation on a set \( A \) with identity element \( e \). Let \( B \) be a subset of \( A \) that is closed under \( * \). Let \( b_1, b_2, \ldots, b_n \in B \). Prove that \( b_1 * b_2 * \ldots * b_n \in B \).

12. Let \( * \) be an associative binary operation on a set \( A \) with identity element \( e \). In Exercise 32 of Section 4.1, you were asked to prove that if \( B \) and \( C \) are subsets of \( A \) that are closed under \( * \), then \( B \cap C \) is closed under \( * \). Prove by induction that if \( B_1, B_2, \ldots, B_n \) are subsets of \( A \) that are all closed under \( * \), then \( B_1 \cap B_2 \cap \ldots \cap B_n \) is closed under \( * \).

13. (a) Let \( n \) be an integer. Prove by induction that if \( n \) is even, then \( n^k \) is even for all \( k \in \mathbb{Z}^+ \).

(Note: the induction should be done with the variable \( k \), not the variable \( n \).)

(b) State the converse of part (a). Prove or disprove.

14. Prove by induction that if \( n_1, n_2, \ldots, n_t \) are even integers, then \( n_1 + n_2 + \ldots + n_t \) is even.

15. Let \( A, B_1, B_2, \ldots, B_n \) be sets. Generalize part 2 of Theorem 2.2.3 by proving that

\[
A \cup (B_1 \cap B_2 \cap \ldots \cap B_n) = (A \cup B_1) \cap (A \cup B_2) \cap \ldots \cap (A \cup B_n).
\]

16. Let \( A \) be a nonempty set. Let \( f_1, f_2, \ldots, f_n \in \mathcal{F}(A) \). Prove:

(a) If \( f_1, f_2, \ldots, f_n \) are surjective, then the composition \( f_1f_2 \ldots f_n \) is surjective.

(b) If \( f_1, f_2, \ldots, f_n \) are injective, then the composition \( f_1f_2 \ldots f_n \) is injective.

(c) If \( f_1, f_2, \ldots, f_n \) are invertible, then the composition \( f_1f_2 \ldots f_n \) is invertible and \( (f_1f_2 \ldots f_n)^{-1} = f_n^{-1}f_{n-1}^{-1} \ldots f_2^{-1}f_1^{-1} \).

17. Let \( P(n) \) be a statement about the positive integer \( n \). Using the negations of conditions 1 and 2 in Theorem 5.2.1, complete the following sentence: \( P(n) \) is false for some positive integer \( n \) if . . . .

18. Prove Theorem 5.2.2.

19. Prove Theorem 5.2.3.

20. State a modified form of the Second Principle of Induction similar to the modified form of the First Principle of Induction.

21. Prove Theorem 5.2.4.
integers less than $10^{10}$. So although just about everyone believes that it is true, no general proof seems forthcoming in the immediate future.

**Exercises 5.4**

1. Prove Lemma 5.4.2.

2. Let $n \in \mathbb{Z}$, $n > 1$. Prove that if $n$ is not divisible by any prime number less than or equal to $\sqrt{n}$, then $n$ is a prime number.

3. Let $n$ be a positive integer greater than 1 with the property that whenever $n$ divides a product $ab$ where $a, b \in \mathbb{Z}$, then $n$ divides $a$ or $n$ divides $b$. Prove that $n$ is a prime number.

4. Prove Corollary 5.4.5.

5. Prove Corollary 5.4.6.

6. (a) Prove that $\sqrt{2}$ is irrational.
   (b) Prove that $\sqrt[3]{2}$ is irrational.
   (c) Prove that $\sqrt[3]{2}$ is irrational for every $n \in \mathbb{Z}, n \geq 2$.
   (d) Prove that if $p$ is a prime number, then $\sqrt{p}$ is irrational for every $n \in \mathbb{Z}, n \geq 2$.
   (e) Let $n, m \in \mathbb{Z}, n \geq 2$. Prove that if $m$ is not the $n$th power of an integer, then $\sqrt[n]{m}$ is irrational.

7. (a) Prove that $\log_{10} 3$ is irrational.
   (b) Prove that if $r$ is a rational number such that $r > 1$ and $r \neq 10^n$ for any positive integer $n$, then $\log_{10} r$ is irrational.

8. Write the following integers in standard form:
   (a) 594
   (b) 1,400
   (c) 42,750
   (d) 191,737

9. Let $n \in \mathbb{Z}, n \geq 1$. Prove that $n$ is a perfect square if and only if, when $n$ is written in standard form, all of the exponents are even.

10. Let $a, b \in \mathbb{Z}$.
    (a) Prove that if $a^2 \mid b^2$, then $a \mid b$.
    (b) Prove that if $a^n \mid b^n$ for some positive integer $n$, then $a \mid b$.

11. (a) Let $a, b \in \mathbb{Z}$ such that $(a, b) = 1$. Suppose that $ab = x^2$ for some $x$ in $\mathbb{Z}$. Prove that $a = y^2$ and $b = z^2$ for some $y$ and $z$ in $\mathbb{Z}$.
    (b) Show that part (a) is false without the assumption that $a$ and $b$ are relatively prime.
    (c) Let $a, b \in \mathbb{Z}$ such that $(a, b) = 1$. Suppose that $ab = x^n$ for some $x$ in $\mathbb{Z}$ and some positive integer $n$. Prove that $a = y^n$ and $b = z^n$ for some $y$ and $z$ in $\mathbb{Z}$.
    (d) Let $a, b, c \in \mathbb{Z}^+$ such that $(a, b) = (a, c) = (b, c) = 1$. Suppose that $abc = x^2$ for some $x$ in $\mathbb{Z}^+$. Prove that $a, b,$ and $c$ are all squares in $\mathbb{Z}^+$. 
5.4 PRIMES AND UNIQUE FACTORIZATION

(e) Let \( a_1, a_2, \ldots, a_n \in \mathbb{Z}^+ \) such that \((a_i, a_j) = 1\) if \(i \neq j\). Suppose that \(a_1a_2 \ldots a_n = x^2\) for some \(x\) in \(\mathbb{Z}^+\). Prove that each \(a_i\) is a square in \(\mathbb{Z}^+\).

12. Prove that if a positive integer of the form \(2^m + 1\) is prime, then \(m\) is a power of 2.

13. Prove that 2 is the only prime of the form \(n^3 + 1\).

14. Prove that if \(2^n - 1\) is prime, then \(n\) is prime.

15. Investigate the following statement:

   If \(n\) is any positive integer, then \(n^2 + n + 41\) is always a prime number.

   If you think it is true, give a proof; if false, give a counterexample.

16. Let \(a, b \in \mathbb{Z}^+, a > 1, b > 1\). Let \(a = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots \cdot p_r^{\alpha_r}\) and \(b = p_1^{\beta_1} \cdot p_2^{\beta_2} \cdots \cdot p_r^{\beta_r}\), where \(p_1, p_2, \ldots, p_r\) are primes and \(\alpha_i\) and \(\beta_i\) are nonnegative integers, for \(i = 1, 2, \ldots, r\). Let \(l_i = \min(m_i, n_i)\). Prove that \((a, b) = p_1^{l_1} \cdot p_2^{l_2} \cdots \cdot p_r^{l_r}\).

17. Use Exercises 8 and 16 to find the greatest common divisor of 1,400 and 42,750.

18. Prove that if \(a\) is a positive integer of the form \(4n + 3\), then at least one prime divisor of \(a\) is of the form \(4n + 3\).

19. Prove that if \(a\) is a positive integer of the form \(3n + 2\), then at least one prime divisor of \(a\) is of the form \(3n + 2\).

20. Prove that there are infinitely many primes of the form \(3n + 2, n \in \mathbb{Z}^+\).

21. Prove that there are infinitely many primes of the form \(6n + 5, n \in \mathbb{Z}^+\).

22. Let \(n\) be a positive integer. Prove that the binomial coefficient \(\binom{2n}{n}\) is divisible by every prime \(p\) such that \(n < p \leq 2n\) but is not divisible by \(p^2\).

23. Let \(p\) be a prime number and \(t\) a positive integer. Let \(a \in \mathbb{Z}\). Suppose that \(a\) divides \(p^t\). Prove that \(a = p^k\) for some \(k \in \mathbb{Z}, 1 \leq k \leq t\).

24. Let \(n, m \in \mathbb{Z}, (n, m) = 1\). Suppose that \(d\) is a positive divisor of \(nm\). Prove that there exist positive integers \(d_1\) and \(d_2\) such that \(d = d_1d_2\) where \(d_1\) divides \(n\) and \(d_2\) divides \(m\).

25. If \(n\) is a positive integer, let \(\tau(n)\) denote the number of positive divisors of \(n\). So, for example, \(\tau(1) = 1, \tau(2) = 2, \tau(3) = 2, \tau(4) = 3, \tau(5) = 2, \tau(6) = 4\).

(a) Prove that if \(p\) is a prime number and \(t\) is a positive integer, then \(\tau(p^t) = t + 1\).

(b) Let \(n, m \in \mathbb{Z}, (n, m) = 1\). Prove that \(\tau(nm) = \tau(n)\tau(m)\).

(c) Let \(n \in \mathbb{Z}\). Let \(n = p_1^{a_1} \cdot p_2^{a_2} \cdots \cdot p_r^{a_r}\) be the prime factorization of \(n\). Prove that \(\tau(n) = (a_1 + 1)(a_2 + 1) \cdots (a_r + 1)\).