Read the following, which can all be found either in the textbook or on the course website.

- Chapter 8.2 of *Visual Group Theory* (VGT).
- VGT Exercises 8.11, 8.13, 8.14, 8.16, 8.17

Write up solutions to the following exercises.

1. Let \( \mathbb{Q} \) be the group of rational numbers under addition, \( \mathbb{Q}^* \) be the group of non-zero rational numbers under multiplication, and let \( \mathbb{Q}^+ \) be the group of positive rational numbers under multiplication.

   (a) Prove that \( C_2 \cong \{1, -1\} \).
   (b) Describe the quotient groups \( \mathbb{Q}/\langle 1 \rangle \) and \( \mathbb{Q}^*/\langle -1 \rangle \). In particular, what do the elements (cosets) look like?
   (c) Show that \( \mathbb{Q}^* \cong \mathbb{Q}^+ \times C_2 \).

2. For Parts (a)–(d), a group \( G \) is given together with a normal subgroup \( H \). Illustrate the embedding \( \phi: H \to G \), and the quotient map \( q: G \to G/H \), chained together so that \( \text{im}(\phi) = \ker(q) \). An example for \( G = \mathbb{Z}_6 \) and \( H = \mathbb{Z}_2 \) is shown below:

   ![Diagram](chart)

   (a) \( G = \mathbb{Z}_6, H = \mathbb{Z}_3 \),
   (b) \( G = S_3, H = C_3 \),
   (c) \( G = A_4, H = V_4 \),
   (d) \( G = S_3, H = A_3 \)

   Now, answer the following question about each of your answers to Parts (a)–(c).

   (e) What group is \( G/H \) isomorphic to? Write out the isomorphism.

3. Let \( A \) and \( B \) be normal subgroups of \( G \). In this problem, you will prove the *Diamond Isomorphism Theorem*.

   (a) Prove that the set \( AB := \{ab : a \in A, b \in B\} \) is a subgroup of \( G \).
   (b) Prove that \( B \triangleleft AB \) and \( A \cap B \triangleleft A \).
(c) Prove that $A/(A \cap B) \cong AB/B$. [Hint: Construct a homomorphism $\phi: A \to AB/B$ that has kernel $A \cap B$, then apply the FHT.]

(d) Draw a diagram, or lattice, of $G$ and its subgroups $AB, A, B,$ and $A \cap B$. Interpret the result in Part (c) in terms of this diagram.

4. For each part below, consider the group $G = \langle A, B \rangle$ generated by the two matrices shown. Assume that matrix multiplication is the binary operation, and $i = \sqrt{-1}$. To what common group is $G$ isomorphic? Write down an explicit isomorphism (you only need to define it for the generators), and a group presentation for $G$.

(a) $A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$

(b) $A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}.$

(c) $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$

5. Let $H \leq G$, and fix $x \in G$. Recall that we showed in class that $xHx^{-1}$ is always a subgroup of $G$.

(a) Prove additionally that $xHx^{-1} \cong H$. [Hint: Define a mapping from $H$ to $xHx^{-1}$ and prove that it is a homomorphism, one-to-one, and onto.]

(b) Use Part (a) to show that $o(xy) = o(yx)$ for any $x, y \in G$. (Recall that $o(g)$ is the order of $g$.)

6. In this exercise, you will prove that if $A$ and $B$ are normal subgroups of $G$, and $AB = G$, then

$$G/(A \cap B) \cong (G/A) \times (G/B).$$

(a) Consider the following map:

$$\phi: AB \rightarrow (G/A) \times (G/B), \quad \phi(g) = (gA, gB).$$

Show that $\phi$ is a homomorphism.

(b) Show that $\phi$ is surjective. That is, given any $(g_1A, g_2B)$, show that there is some $g = ab \in AB$ such that $\phi(g) = (g_1A, g_2B)$. [Hint: Try $g = a_2b_1$.]

(c) Find $\ker(\phi)$ [you need to prove your answer is correct] and then apply the Fundamental Homomorphism Theorem.