Read the following, which can all be found either in the textbook or on the course website.

- Chapters 8.3, 8.4, 8.5 of *Visual Group Theory* (VGT).
- VGT Exercises 8.19, 8.22, 8.23, 8.26, 8.37(ab), 8.41, 8.43-8.50.

Write up solutions to the following exercises.

1. We say that a product of cyclic groups $C_{n_1} \times \cdots \times C_{n_r}$ is “organized by elementary divisors” if $n_i$ divides $n_{i+1}$ for every $i$. For example:

$$C_2 \times C_4 \times C_3 \times C_9 \times C_5 \cong C_9 \times C_{180};$$

the left group is *not* organized by elementary divisors, but the right group is.

For each order given below, list all abelian groups of that order by writing each one as a product of cyclic groups of prime power order. Additionally, write each one as a product of cyclic groups organized by “elementary divisors.”

(a) 8
(b) 54
(c) 400
(d) $p^2q$, where $p$ and $q$ are distinct primes.

2. The commutator subgroup of a group $G$ is the subgroup

$$G' = \langle aba^{-1}b^{-1} \mid a, b \in G \rangle.$$ 

(a) Prove that $G$ is abelian if and only if $G' = \{e\}$.

(b) Prove that $G'$ is the intersection of all normal subgroups of $G$ that contain the set $C := \{aba^{-1}b^{-1} \mid a, b \in G\}$:

$$G' = \bigcap_{C \subseteq N \triangleleft G} N$$

(c) If we quotient $G$ by $G'$, then we are in essence, “killing” all non-abelian parts of the Cayley diagram, as shown below:

Prove algebraically that $G/G'$ is indeed abelian.
3. Find the commutator subgroup of each of the following groups and compute its abelianization.

(a) An abelian group \( A \).
(b) \( Q_4 \)
(c) The alternating group \( A_n \), for \( n \geq 5 \). [\textit{Hint}: \( A_n \) is a \textit{simple group}, which means its only normal subgroups are \( \langle e \rangle \) and \( A_n \).]
(d) The dihedral group \( D_n \) for \( n \) even.
(e) The dihedral group \( D_n \) for \( n \) odd.

4. For each group \( G \), find all automorphisms and make a multiplication table of \( \text{Aut}(G) \). What group is it isomorphic to?

(a) \( \mathbb{Z}_7 \)
(b) \( \mathbb{Z}_8 \)
(c) \( \mathbb{Z}_{10} \)
(d) \( V_4 \)
(e) \( D_3 \)
(f) \( \mathbb{Z}_2 \times \mathbb{Z}_3 \)
(g) \( \mathbb{Z}_2 \times \mathbb{Z}_2 \)