Read the following, which can all be found either in the textbook or on the course website.

- Chapters 8.6, 9.1, 9.2 of *Visual Group Theory* (VGT).

Write up solutions to the following exercises. In the exercises below, the set $\text{Fix } \phi$ of fixed points of a group action $\phi: G \to \text{Perm } S$ is the set of $s \in S$ such that $\text{Stab } s = G$. (I.e. the set of $s \in S$ such that $\phi_g(s) = s$ for all $g \in G$.)

1. Let $G$ act on a set $S$. Prove that $\text{Stab}(s)$ is a subgroup of $G$ for every $s \in S$.

2. If $C_5$ acts on the set $S = \{A, B, C, D\}$, what will the action diagram be? Why?

3. Let $S$ be the following set of 7 “binary squares”:

$$S = \left\{ \begin{array}{ccc}
0 & 0 & 0 \\
0 & 1 & 0 \\
1 & 0 & 1 \\
1 & 1 & 0 \\
0 & 1 & 1 \\
0 & 0 & 1 \\
1 & 0 & 1 \\
\end{array} \right\}$$

   (a) Consider the (right) action of the group $G = V_4 = \langle v, h \rangle$ on $S$, where $\phi(v)$ reflects each square vertically, and $\phi(h)$ reflects each square horizontally. Draw an action diagram and compute the stabilizer of each element.

   (b) Consider the (right) action of the group $G = C_4 = \langle r \mid r^4 = e \rangle$ on $S$, where $\phi(r)$ rotates each square $90^\circ$ clockwise. Draw an action diagram and compute the stabilizer of each element.

   (c) Suppose a group $G$ of size 15 acts on $S$. Prove that there must be a fixed point.

4. Let $G = S_4$ act on itself by conjugation via the homomorphism

$$\phi: G \to \text{Perm}(S), \quad \phi(g) = \text{the permutation that sends each } x \mapsto g^{-1}xg.$$ 

   (a) How many orbits are there? Describe them as specifically as you can.

   (b) Find the orbit and the stabilizer of the following elements:

   i. $e$
   ii. $(1\;2)$
   iii. $(1\;2\;3)$
   iv. $(1\;2\;3\;4)$

5. A *$p$-group* is a group of order $p^k$ for some integer $k$. Recall that the *center* of a group $G$ is the set of all elements that commute with everything:

$$Z(G) = \{ z \in G \mid gz = zg, \forall g \in G \} = \{ z \in G \mid g^{-1}zg = z, \forall g \in G \}.$$

Finally, a group $G$ is *simple* if its only normal subgroups are $G$ and $\langle e \rangle$. 
(a) Let $G$ act on itself by conjugation via the homomorphism
\[ \phi: G \rightarrow \text{Perm}(S), \quad \phi(g) = \text{the permutation that sends each } x \mapsto g^{-1}xg. \]
Prove that $\text{Fix}(\phi) = Z(G)$.
(b) Prove that if $G$ is a $p$-group, then $|Z(G)| > 1$. [Hint: Revisit the Class Equation.]
(c) Use the result of the previous part to classify all simple $p$-groups.

(a) If all subgroups of $G$ of order 4 are isomorphic to $V_4$, then what group must $G$ be? Completely justify your answer.
(b) Next, suppose that $G$ has a subgroup $H \cong C_4$. Then $G$ has a Cayley diagram like one of the following:

```
(a)  a  b  c  d
      |   |
     b   c
```
```
(b)  a  b  c  d
      |   |
     d   b
```

Find all possibilities for finishing the Cayley diagram.
(c) Label each completed Cayley diagram by isomorphism type. Justify your answer.
(d) Make a complete list of all groups of order 8, up to isomorphism.

7. Recall that a group $G$ is called simple if its only normal subgroups are $G$ and $\{e\}$. Let $p$ be a prime.
(a) Show that there is no simple group of order $2p$. (Hint: use the Cauchy’s Theorem)
(b) Show that there is no simple group of order $2p^n$. (Hint: use the 1st Sylow Theorem)