1. Define the complex projective line $\mathbb{P}^1(\mathbb{C})$ to be the set of pairs $(w_1 : w_2)$ with $w_1, w_2$ complex numbers not both 0, modulo multiplication by nonzero complex numbers $\lambda$ (i.e., for $\lambda \in \mathbb{C}^* = \mathbb{C} - \{0\}$, we identify $(w_1 : w_2)$ and $(\lambda w_1, \lambda w_2)$). Define the complex upper half plane $\mathcal{H} := \{a + bi \in \mathbb{C} : b > 0\}$ to be the set of all complex numbers with positive imaginary part.

a) Show that $\mathbb{P}^1(\mathbb{C})$ can be identified with the set of lines through 0 in $\mathbb{C}^2$.

b) Read the definition of “group action” on Wikipedia. Show that the formula

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (w_1 : w_2) = (aw_1 + bw_2 : cw_1 + dw_2)$$

defines an action of $\text{SL}_2(\mathbb{C})$ on $\mathbb{C}^2$, and moreover gives an action on $\mathbb{P}^1(\mathbb{C})$ (identified as the set of lines in $\mathbb{C}^2$ through the origin).

c) Show that $\mathbb{P}^1(\mathbb{C})$ can be identified with $\mathbb{C} \cup \infty$ by mapping $(w_1 : w_2)$ to $w_1/w_2$.

d) Show that this induces an action of $\text{SL}_2(\mathbb{C})$ on $\mathbb{C} \cup \infty$ given by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}(\tau) = (a \tau + b)/(c \tau + d)$ for $\tau \in \mathbb{C} \cup \infty$.

e) Show that this restricts to an action of $\text{SL}_2(\mathbb{R})$ on the upper half plane $\mathcal{H}$, given by the same formula. (In other words, show that the above action preserves $\mathcal{H}$ when the matrix has coefficients in $\mathbb{R}$.)

f) Show that $\Im((a \tau + b)/(c \tau + d)) = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix}|c \tau + d|^{-2}\Im(\tau)$ for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(\mathbb{R})$ (where $\Im(\tau)$ is the imaginary part of $\tau$).