STACKS HW2 - LIMITS, COLIMITS, GROUP OBJECTS

(1) Let C be a category and let $h\colon C \to \mathrm{Fun}(C^{\mathrm{op}}, \mathbf{Sets})$ be the Yoneda embedding. Show that for any arrows $X \to Y$ and $Z \to Y$ in C, there is a natural isomorphism

$$h_{X \times_Y Z} \to h_X \times_{h_Y} h_Z$$

of functors, where $h_X \times_{h_Y} h_Z$ is the functor

$$h_X \times_{h_Y} h_Z \colon W \mapsto h_X(W) \times_{h_Y(W)} h_Z(W).$$

(2) Let G be an object of a category C. Show that the functor of points

$$h_G \colon C^{\mathrm{op}} \to \mathbf{Sets}$$

factors through the forgetful functor from groups

$$\begin{array}{ccc} & & \mathbf{Groups} \\ & \nearrow & \downarrow \\ h_G\colon C^{\mathrm{op}} & \longrightarrow & \mathbf{Sets} \end{array}$$

if and only if G is a group object of C.

(3) Let $\star$ be one of $\operatorname{Spec} k$ (with k a field) or $\operatorname{Spec} \mathbb{Z}$. Let

$$G = \coprod_{\mathrm{g} \in \mathbb{Z}/2\mathbb{Z}} \star$$

be the group object corresponding to $\mathbb{Z}/2\mathbb{Z}$. Work out explictily the group structure. In other words, work out the maps in terms of rings, and show that G represents the functor

$$\mathbf{Sch}^{\mathrm{op}} \to \mathbf{Groups},\ W \mapsto \mathbb{Z}/2\mathbb{Z}$$

(4) (a) Let $n \geq 1$ be an integer and let

$$\mathrm{GL}_n \colon (\mathbf{Sch})^{\mathrm{op}} \to \mathbf{Sets}$$

be the functor sending a scheme Y to the set $\mathrm{GL}_n(\Gamma(Y, \mathcal{O}_Y))$. Prove that GL_n is a representable functor.

(b) Let X represent the functor GL_n. Prove that the group structures on the sets $\mathrm{GL}_n(\Gamma(Y, \mathcal{O}_Y))$ induce the structure of a group scheme on X. (I.e. use the previous exercise, noting that there is one detail to check.)

(5) Give an example of a category C, a subcategory C', and a diagram $D\colon I \to C'$ such that the limit (or colimit, your choice) in C' is not the limit in C.

(6) Use the functor of points to define a map $\mathbb{A}^1 \to \mathbb{A}^1$ given by the formula $z \mapsto z^2$. Compare this with how one would define such a map with locally ringed spaces.

(7) **Monomorphisms**.

(a) Show that a morphism $X \to Y$ is a monomorphism if and only if for every $T \in C$, the map of sets $X(T) \to Y(T)$ is injective.

(b) Show that a map of schemes which is injective topologically may not be a monomorphism.

(c) Show that a map of schemes which is surjective topologically may not be an epimorphism.