Three-dimensional numerical modelling and inversion of magnetometric resistivity data

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Summary

We develop an algorithm to model magnetometric resistivity (MMR) response over an arbitrary 3-D conductivity structure and a method for inverting surface MMR data to recover a 3-D distribution of conductivity contrast. In the forward modeling algorithm, the second-order partial differential equations for the scalar and vector potentials are discretized on a staggered gird using the finite-volume technique. The resultant matrix equations are consequently solved using the bi-conjugate gradient stablizing (BiCGSTAB), combined with the symmetric successive over relaxation (SSOR) preconditioning. In the inversion method, we discretize the 3-D model into a large number of rectangular cells of constant conductivity, and the final solution is obtained by minimizing a global objective function composed of the model objective function and data misfit. Since 1-D conductivity variations are an annihilator for surface MMR data, the model objective function uses relative conductivity with respect to a reference model. A depth weighting that counteracts the natural decay of the kernels is incorporated. All minimizations are carried out with the Gauss-Newton algorithm and model perturbations at each iteration are obtained by a conjugate gradient least squares method (CGLS), in which only the sensitivity matrix and its transpose multiplying a vector are required. Through synthetic model studies, we have shown that there are two forms of fundamental ambiguities for recovery of conductivity. For a body buried in a uniform host medium, we can determine only the relative conductivity contrast, not the absolute values. The choice of a constant reference model has no effect on the reconstruction of the relative conductivity. For a 3-D body in a 1-D layered medium, there is additional ambiguity in determining conductivities of the 1-D medium and the target, if receivers are at the surface. A favorable scenario is when the overburden is thin and moderately conductive (less than 10 times the conductivity of underlying basement). Multi-sources and multi-component data will greatly improve the inversion result. We demonstrate our inversion method with tests on both synthetic examples and on a field data set.

Key words: magnetometric resistivity, three-dimensional, forward modelling, inversion, MMR

1 Introduction

The magnetometric resistivity (MMR) method involves measurement of magnetic fields associated with artificially created, non-inductive (DC or pseudo-DC) current flow input to the earth through two electrodes. Historically, for a surface MMR survey, the wire connecting the two current electrodes is typically laid in a horseshoe array and measurements are made somewhere in between the electrode spread. The observations are then reduced by subtracting the theoretical magnetic fields due to current flow in the wire and current flow in a uniform half-space or 1-D layered earth. The latter is often referred as to "normal" magnetic field. After the reduction, the remainders are expressed as a percentage of the normal field at either the center of the study area, or center of the relevant profile. Information about the conductivity distribution beneath the surface is then extracted with the aid of some data processing techniques. For down-hole or marine MMR measurements, data processing is slightly different since receivers are located in bore-holes or on the seafloor. Edwards & Nabighian (1991) gives a thorough description of the theoretical work, modern implementation, and case studies about the MMR method. In this paper, though, we do not explicitly connect the words MMR to a specific layout of current electrodes. The work here is valid for any location of current sources and receiver location. We will however use the typical houseshoe layout as an example since it serves as a good illustration and also is needed for our field data set.

MMR methods have advantages over conventional electric resistivity methods. The MMR measurement has greater sensitivity to conductive targets beneath a moderately conductive overburden than does the DC method (Edwards & Nabighian, 1991). For example, an overburden of conductivity 0.01 S/m, 30 m thick, will dramatically reduce the response from a conductive target located in a basement of 0.001 S/m for a DC survey; however, the corresponding MMR anomaly is only reduced by about 10%, compared with that without the conductive overburden (see Fig. 11). This makes DC measurements difficult to apply in such areas where the weathered layer has conductivity one or more orders of magnitude higher than the conductivity of the underlying rock, but the MMR a feasible method. In addition, MMR is only sensitive to the relative conductivity between the targets and their surroundings, not the absolute conductivity values. This makes MMR an attractive technique for detecting poorly conducting targets such as some zinc deposits (Bishop *et al*., 1997).

Despite these positive attributes, MMR is not a routinely used method in mineral exploration although it was successfully applied by Edwards (1974) two decades ago. There are two principal difficulties. The first is that the MMR signal is generally low and great care and good instrumentation are required at the data acquisition phase. The second impediment is the complexity of the observations; the maps of individual components bear no simple relationship to the geometry of the conductors. Fig. 1 is a manifestion of this. A simple cubic body with side length of 400 m, conductivity 0.1 S/m, is located in a host of 0.001 S/m. The top depth of the cube is 80 m. The source and sink electrodes are located on the x-axis; each is 600 m from the origin of the coordinates. The synthetic three components of the anomalous magnetic field at the surface have totally different anomaly patterns; the y-component is positive above the target; the vertical component changes sign from the upper panel to the bottom, while x-component shows an anomaly in all four quadrants. Except for y-component, there is little compelling evidence to indicate a conductive body at depth, nevermind trying to directly infer information about the conductivity contrast and buried depth.

Calculation of MMR responses over a conductive structure can be roughly grouped into three categories. The first is based on analytical derivations for simple structures. As summarized in Edwards *et al*. (1978), these structures include an anisotropic earth, vertical and dipping contacts, thin and thick dykes, and semicylindrical and hemispherical depressions, as well as α-media. Inayat-Hussain (1989) also developed an analytical algorithm to compute the magnetic field of a direct current in a cylindical-shaped conductor imbedded in a resistive half-space beneath a conductive surface layer by using a Fourier series integral. The second method for forward modelling is based upon the modification of a numerical "resistivity" method (Edwards & Nabighian, 1991). This involves two steps. The first is to solve Poisson's equation for the electric potential by using the standard finite element or finite difference techniques. This is the same procedure implemented in the conventional DC forward problem. The second step is to calculate the magnetic field through the modified form of the Biot-Savart law in which magnetic field is explicitly expressed as a volume integral of a functional that is proportional to the cross product of gradient of the potential and conductivity throughout the volume (Edwards *et al*., 1978). Pai and Edwards (1983), Acosta & Worthington (1983), Yang & Tseng (1992) all followed this procedure to compute MMR response over a 2-D conductivity model. More recently, Boggs *et al*. (1999) developed a finite difference method for evaluating total field magnetometric resistivity (TFMMR) responses of 3-D structures. In their work, the magnetic field was evaluated in the Fourier domain. The third method for forward modelling is the surface integral equation method. For many simple problems, the gradient of the electrical conductivity vanishes everywhere except on the surfaces defining changes in conductivity. Consequently, the volume integral for the modified Biot-Savart law is reduced to a finite set of surface integrals. Gomez-Trevino & Edwards (1979) derived a rapid algorithm for evaluating the three components of the magnetic field over a 2-D structure. Oppliger (1984) modeled the effect of 2-D topography, while Nabighian *et al*. (1984), and Cheesman & Edwards (1989) computed MMR anomalies associated with multiple finite plates of arbitrary conductance.

In this paper we perform forward modelling by using the method proposed by Haber (2000) except we use a finite volume solution rather than the mixed finite elements. Effectively the MMR modelling can be regarded as a mixed sub-problem of electrostatic and magnetostatic problems. One first solves an electrostatic problem for a scalar potential, and then solves a magnetostatic equation for magnetic field. The first stage is therefore similar to that proposed in the previous paragraph, but the second stage is different in that we obtain the magnetic field by solving a differential equation rather than by performing a volume integration.

In contrast to the forward modelling, inversion techniques of MMR data are less developed. Type curves (Howland-Rose *et al*., 1980) are used to qualitatively estimate the source of some simple 2-D targets and trial and error interpretation has been developed based upon the gravity-MMR relationship given by Szarka (1987). This relationship has been exploited with some success by using the standard 2-D gravity modelling suite for processing down-hole MMR data (Asten, 1988). As recognized by Bishop *et al*. (1997), however, standard gravity programs, which generally assume flat surface traverses, may be not suited to surveys down deviating drill holes. Consequently, the results have to be best-fitted by hand onto a geological cross-sections. A full 3-D interpretation program is expected to be able to overcome such difficulties and to extract more useful information about the conductivity from the measured data.

In this paper, we first develop a numerical algorithm to model the MMR responses of an arbitrary 3-D conductivity structure. The governing second-order differential equations for scalar and vector potentials are split into a couple of first-order equations, and then discretized on a staggered grid by using a finite-volume method (Haber & Ascher, 2001). This discretization scheme is second-order accurate and allows us to cope with highly discontinuous conductivity and permeability. Scalar potentials are defined at the centers of grid cells while vector potentials are located on the faces. Conductivity values at cell faces are harmonically averaged while permeability values at edges are arithmetically averaged. The resulting matrix equations are solved using the biconjugate gradient stablized (BiCGSTAB) method, combined with a symmetric successive over relaxation (SSOR) preconditioning. Once the two potentials are solved, the magnetic field (and electric field, if required) can be obtained anywhere by applying the discrete matrix for the curl operator to the vector potentials. This numerical algorithm is verified by comparison with the analytical solutions over a vertical contact and a hemispherical depression. We also discuss the relationship between the MMR anomaly and the conductivity distribution; this helps us define the "model" parameter in our inversion algorithm. The inverse problem for MMR data is formulized as an optimization problem in which we minimize a model objective function subject to the constraints that the data misfit is achieved to some level. Our model objective function has the flexibility to incorporate extra information, and a Gauss-Newton iterative method is used to obtain the model perturbation at each iteration. The regularization parameter that controls the balance between model norm and misfit is determined through a cooling process. At each iteration we must solve a large matrix system. We use a conjugate gradient least-squares (CGLS) method and hence the majority of computations involve multiplying a sensitivity matrix, or its transpose, by an arbitrary vector. This can be accomplished without explicitly calculating and storing the sensitivity matrix (Haber *et al*., 2000b).

The paper begins by formulating the finite-volume discretization for forward modelling of MMR response and then by formulating the inverse algorithm. Practical considerations pertaining to the inverse problems, such as how to define a depth weighting to counteract to the natural decay of the kernels, how to choose a reference model, and which component data to be inverted will be discussed through synthetic data investigation. The paper concludes by inverting a field data set over a known mineral deposit and a subsequent conclusion.

2 3-D MMR Forward modelling

2.1 Governing equations

As shown in Fig. 2, an external current is impressed into the ground through a pair of current electrodes C1 and C2. The magnetic field associated with the current flow in both the wire and ground can be measured by a magnetometer at the surface. For MMR modelling we can assume that the exciting source is a direct current (DC). This means the solution for general Maxwell's equations in the frequency domain can be reduced to a steady-state problem (frequency is zero), which can be written as

$$\nabla \times \mathbf{E} = 0 , \tag{1a}$$

$$\nabla \times \mathbf{H} - \sigma \mathbf{E} = \mathbf{J}^s , \tag{1b}$$

$$\nabla \cdot (\mu \mathbf{H}) = 0 , \tag{1c}$$

where $\mathbf{E}$ is the electric field intensity in V/m, $\mathbf{H}$ is the magnetic field intensity in A/m, $\mathbf{J}^s$ is the external electric current density in A/m^2, σ and μ are the electric conductivity and magnetic permeability, respectively. The constitutive relation $\mathbf{J} = \sigma \mathbf{E}$ has been incorporated into eq.(1b).

From eq. (1a), there exists a scalar electric potential ϕ, which allows us to write $\mathbf{E}$ as

$$\mathbf{E} = -\nabla \phi . \tag{2}$$

By taking divergence of eq. (1b), we obtain the well-known dc equation

$$\nabla \cdot (\sigma \nabla \phi) = \nabla \cdot \mathbf{J}^s . \tag{3}$$

With appropriate boundary conditions, this system of equations can be solved for ϕ. Once the potential ϕ is obtained, the magnetic field $\mathbf{H}$ can be computed by solving the following equations

$$\nabla \times \mathbf{H} = \mathbf{J}^s - \sigma \nabla \phi = \mathbf{f} , \tag{4a}$$

and

$$\nabla \cdot (\mu \mathbf{H}) = 0 , \tag{4b}$$

where the scalar potential ϕ is on the right side of eq.(4a), and serves as a source term to produce the magnetic field. The source term $\mathbf{f}$ in eq. (4a) should implicitly satisfy the compatibility condition $\nabla \cdot \mathbf{f} = 0$. Eq. (4), with boundary conditions, defines a magnetostatic problem, and many numerical methods have been devoted to solving it (Jin, 1993; Haber, 2000). In order to solve for $\mathbf{H}$, we introduce a magnetic potential $\mathbf{A}$ such that

$$\mu \mathbf{H} = \nabla \times \mathbf{A} . \tag{5}$$

The introduction of the vector potential $\mathbf{A}$ makes eq. (4b) implicitly satisfied. Therefore, the system of equations for $\mathbf{A}$ can be written as

$$\nabla \times \mu^{-1} \nabla \times \mathbf{A} = \mathbf{f} , \tag{6a}$$

subject to appropriate boundary conditions. As encountered in other electromagnetic problems (Jin, 1993), this equation does not have a unique solution due to the null space of the curl operator, and therefore a gauge condition for $\mathbf{A}$ has to be imposed. We adopt the Coulomb gauge condition (Haber *et al*., 2000a)

$$\nabla \cdot \mathbf{A} = 0 \quad . \tag{6b}$$

The system of eqs (6) is in principle over-determined, since we have 4 equations but only 3 unknowns. However, it is consistent since eq. (6b) exactly covers the null-space of the operator in eq. (6a). To ensure that the system remains positive definite, a stabilizer is added (Haber & Ascher, 2001)

$$\nabla \times \mu^{-1} \nabla \times \mathbf{A} - \nabla \mu^{-1} \nabla \cdot \mathbf{A} = \mathbf{f} \quad . \tag{7}$$

In order to obtain a second-order accurate method, even in the case of highly discontinuous conductivity and susceptibility, the variable

$$\psi = \mu^{-1} \nabla \cdot \mathbf{A} \ , \tag{8}$$

is introduced and $\mathbf{H}$ is not eliminated from the system. The final system becomes

$$\nabla \times \mathbf{A} - \mu \mathbf{H} = 0 \, , \tag{9a}$$

$$\nabla \cdot \mathbf{A} - \mu \psi = 0 \ , \tag{9b}$$

$$\nabla \times \mathbf{H} - \nabla \psi = \mathbf{f} \quad , \tag{9c}$$

which leads to a first-order, mixed formulation for the unknowns ($\mathbf{A}$, $\mathbf{H}$, ψ). Similarly, eq. (3) can also be decoupled into two first-order systems for the unknowns ($\mathbf{J}$, ϕ) as

$$\nabla \cdot \mathbf{J} = -\nabla \cdot \mathbf{J}^{s} \ , \tag{10a}$$

$$\sigma^{-1} \mathbf{J} - \nabla \phi = 0 \ . \tag{10b}$$

2.2 Discretization using finite volumes

The differential eqs (9) and (10) are discretized by a finite-volume method on a staggered grid. An appealing characteristic of the finite-volume method is that discontinuous fields, such as the normal component of the electric field separating two regions of different conductivities, are handled by working with fluxes that are continuous. The study region is divided up by three orthogonal sets of constant co-ordinate surfaces, producing a matrix of rectangular cells. The 3-D volume must include the air since we want to solve for $\mathbf{A}$ and ϕ by using the same grid mesh. Each grid cell is assumed to have constant material properties (conductivity and permeability), but the property values can significantly vary from one cell to the next. In formulating a solution, care must be taken to consider the smoothness properties of the different unknown variables. Inappropriate location of variables can violate the conservation laws and implicit boundary conditions, and result in erroneous solutions. In our discretization scheme, $\mathbf{A}$ and $\mathbf{J}$ are chosen to be at centers of cell faces, $\mathbf{H}$ at centers of cell edges and ψ and ϕ are at cell centers, as shown in Fig. 3. By doing so, the continuity of normal $\mathbf{J}$, and tangential $\mathbf{H}$ across boundaries is explicitly preserved.

After having carefully defined the locations of variables, we can discretize eqs (9) and (10) in their weak forms. The detail derivations are given in Haber and Asher (2001). It is worthwhile to point out that firstly, this discretization is second-order accurate for both potentials and $\mathbf{H}$. Secondly, we use harmonic averaging for values of σ at cell faces, and use arithmetic averaging for values of μ at cell edges. These choices arise naturally from the discretization of eqs (9) and (10), and are common to mixed methods. Thirdly, we can eliminate the auxiliary components of $\mathbf{J}$, $\mathbf{H}$, and ψ unambiguously by substituting eq. (10b) into (10a), (9a) and (9b) into (9c), respectively. These algebraic eliminations correspond exactly to discretizing the second-order systems (3) and (7). Our resultant matrix system is

$$\begin{pmatrix} \nabla_h^{(e)} \times \mathbf{M}_e^{-1} \nabla_h^{(f)} \times \; - \nabla_h \mathbf{M}_c^{-1} \nabla_h \cdot & \mathbf{S}\nabla_h \\ 0 & \nabla_h \cdot \mathbf{S}\nabla_h \end{pmatrix} \begin{pmatrix} \mathbf{A} \\ \phi \end{pmatrix} = \begin{pmatrix} \mathbf{J}^s \\ \nabla_h \cdot \mathbf{J}^s \end{pmatrix}, \tag{11}$$

where the matrices $\nabla_h^{(e)} \times$ and $\nabla_h^{(f)} \times$ are assembled from the discretization of the **curl** operator, projecting from cell edges to faces and from faces to edges, respectively; matrices $\nabla_h \cdot$ and ∇_h correspond to the discretization of the **div** and **grad** operators. The material matrix $\mathbf{S}$ arises from the discretization of the conductivity, that is a harmonic averaging of values at cell faces. The matrices $\mathbf{M}_e$ and $\mathbf{M}_c$ result from an arithmetic averaging of permeability at cell edges and the permeability at the cell centers, respectively. The superscript –1 represents the inverse of the matrix.

As pointed out by many researchers, such as Zhang *et al*. (1995) and Zhao & Yedlin (1996), boundary conditions are critical to accurately model the 3-D DC response by using finite-difference method. Mixed boundary conditions (Dey & Morrison, 1979) are often used. In our implementation of finite-volume method, we do not impose explicitly boundary conditions on scalar potential ϕ, but normal components of **A** and **J**, and the tangential components of **H** are assumed to be zero. This requires that the outer boundaries must be far from the electrodes. In our following synthetic examples, the outer boundary in each direction is 2 km away from the origin, including the top boundary in the air.

2.3 Solving the system of equations

Solution of the large system of equations that result from a finite-volume approximation over a 3-D mesh can be solved by using the Krylov space iterative methods because direct solution requires prohibitive amounts of memory and computation. The convergence of Krylov space methods depends on the condition number of the system matrix and on the clustering property of the eigenvalues. Convergence is greatly accelerated by applying an appropriate preconditioner. We use biconjugate gradient stabilized method, or BiCGSTAB, proposed by van der Vorst (1992), combined with a symmetric successive over relaxation (SSOR) for our problem.

Obviously, the system of eq. (11) is decoupled and it is not necessary to solve for $\mathbf{A}$ and ϕ simultaneously. We split them into two systems

$$\nabla_h \cdot \mathbf{S}\nabla_h\phi = \nabla_h \cdot \mathbf{J^s} \quad , \tag{12}$$

and

$$(\nabla_h^{(e)} \times \mathbf{M}_e^{-1}\nabla_h^{(f)} \times \; - \nabla_h \mathbf{M}_c^{-1}\nabla_h \cdot)\mathbf{A} = \mathbf{J^s} - \mathbf{S}\nabla_h\phi \quad . \tag{13}$$

The matrix eq. (12) is first solved, and we then substitute ϕ into eq. (13) and solve for $\mathbf{A}$. A standard left and right preconditioning in the SSOR is applied to eqs (12) and (13), and then we use the BiCGSTAB subroutine coded by Barrett *et al*. (1994), to solve for ϕ and $\mathbf{A}$.

After $\mathbf{A}$ and ϕ are obtained, the fields $\mathbf{H}$ (or $\mathbf{B}$) and $\mathbf{E}$ (if required) can be computed elsewhere by matrix operations $\mathbf{M}_e^{-1}\nabla_h^{(f)} \times \mathbf{A}$ (or $\nabla_h^{(f)} \times \mathbf{A}$ for $\mathbf{B}$) and $-\nabla_h\phi$. For convenience, we do not distinguish $\mathbf{H}$ and $\mathbf{B}$ from now on. Both are called as the magnetic field.

2.4 Computing magnetic data for MMR surveys

In a general MMR survey, as shown in Fig. 2 for a typical surface set-up, a magnetic field sensor measures a component of the magnetic field that exists at any location. The magnetic field is due to currents that flow in the earth ($\mathbf{B}^g$) and in the wire ($\mathbf{B}^w$). Without consideration of various noise, the observed magnetic field $\mathbf{B}^{obs}$ can be denoted by

$$\mathbf{B}^{obs} = \mathbf{B}^w + \mathbf{B}^g \quad . \tag{14}$$

From the numerical view, $\mathbf{B}^w$ and $\mathbf{B}^g$ can be computed separately from the discretized matrix equations as shown in eq. (13). The right side of eq. (13) is the current source terms. The first term $\mathbf{J}^s$ corresponds to the current flow in the wire; the term $-\mathbf{S}\nabla_h\phi$ represents the current flow in the ground. This means we have two options for calculating the magnetic field. Eq. (13) could be solved by including both current source terms. This is feasible but there are substantial discretization errors when fields are to be calculated close to the wire. This arises, because in the numerical implementation of a finite-volume solution, the current in the wire is effectively distributed over the face of a cell. A better procedure, and one used here, is to leave $\mathbf{J}^s$ out of eq. (13) and compute the magnetic field $\mathbf{B}^w$ analytically using the Biot-Savart law. The magnetic field $\mathbf{B}^w$ about a finite straight length of wire carrying current I can be obtained from the Biot-Savart law and is

$$\mathbf{B}^w = \frac{\mu I}{4\pi r}(\cos\alpha - \cos\beta)(\hat{\mathbf{I}} \times \hat{\mathbf{r}}) \quad , \tag{15}$$

where $\hat{\mathbf{I}}$ is a unit vector in the direction of current flow, and $\hat{\mathbf{r}}$ is a unit vector perpendicular to the wire and on the surface of the earth (see Fig. 4). It is evident that $\mathbf{B}^w$ does not contain any information about the conductivity.

Following the MMR convention, the ground component $\mathbf{B}^g$ can also be split into two parts:

$$\mathbf{B}^g = \mathbf{B}^n + \mathbf{B}^a \quad , \tag{16}$$

where $\mathbf{B}^n$ is the normal magnetic field associated with the current flow in a background media which may be a uniform half-space or 1-D earth; $\mathbf{B}^a$ is the anomalous magnetic field which is produced by the current variation on the basis of the normal current. As shown in (Edwards & Nabighian, 1991), $\mathbf{B}^n$ at the surface is independent of the 1-D conductivity distribution beneath the earth, and is only related to the current amplitude and distance from the current. The vertical component of $\mathbf{B}^n$ is zero and the azimuthal $\mathbf{B}^n$, due to a single impressed source electrode with a current of strength I over any 1-D earth, is given by

$$B_\phi^n = \frac{\mu I}{4\pi r} \quad , \tag{17}$$

where r is the distance from the observation point to the electrode. This expression is independent of conductivity. It should be noted that eq. (17) only applies for surface MMR without any topography.

From the above, it follows that the only portion of useful signal from which electrical properties can be inferred is the anomalous magnetic field

$$\mathbf{B}^a = \mathbf{B}^{obs} - \mathbf{B}^w - \mathbf{B}^n \quad . \tag{18}$$

Our numerical tests will focus on this quantity.

2.5 Verification of the code

Edwards *et al*. (1978) present some analytical solutions for vertical and dipping contacts, thin and thick dikes, hemi-cylindrical, and hemi-spherical depressions. We checked our code with two models: a vertical contact and an extremely conductive hemi-spherical depression. Analytical models were calculated directly from formulae given in Boggs (1999), which are slightly different from those given in Edwards *et al*. (1978). The discrepancies are attributed to typographical errors in the original reference (Boggs *et al*., 1999). Only formulae for the vertical component of the anomalous magnetic field, B_z^a, were available so we limited our comparison to that component.

As shown in Fig. 5c, the vertical contact model consisted of two adjacent quarter-spaces with conductivities 0.01 S/m and 0.001 S/m. The contact separating the two spaces is coincident with the *yoz* plane and the source and sink electrodes were located at the surface at the positions (0, -600, 0) m and (0, 600, 0) m, respectively. The analytical expression for B_z^a (Edwards & Nabighian, 1991, p.63) is not reproduced here.

For numerical modelling, the 3-D model, 4 km x 4 km x 4 km, was un-evenly discretized into 54x50x44 cells, including the air layer of 2 km. A mesh size of 25 m was used to partition the center region of the model and a larger mesh size for the rest of model. The vertical component of anomalous magnetic field over the area bounded by the coordinates –400 m to 400 m in both x and y directions, is shown in Fig. 5a. There is little visual difference between the analytical and computed magnetic field, so to compare, we plot the relative error. Specifically, as used in Boggs *et* *al*. (1999), we plot the difference divided by the peak value (~ 450 pT) of the analytical results within the computational area. Fig. 5b is the percentage error.

In a second comparison we modelled responses from a conductive (10 S/m) hemi-sphere of radius 200 m located at the origin of the coordinates. The host medium had a conductivity of 0.001 S/m and current flow was from the source at (0,-600, 0) m to the sink at (0, 600, 0) m. The analytical vertical magnetic fields are given in Edwards & Nabighian (1991, p. 71).

We used the same mesh as in the contact case to discretize this hemisphere model. Results of the 3-D numerical modelling are presented in Fig. 6a, and percentage error in the model is shown in Fig. 6b. Larger errors (up to 8%) occur at the rim of the hemisphere but errors in the rest of the study area are quite small. The error at the rim of the hemisphere is principally due to the difficulty in trying to represent a spherical surface with cuboidal prisms. Rerunning the forward modelling after decreasing the cell size by a factor of 2 at the core area reduced the error to 4.6%.

2.6 Relationship between B and conductivity

Understanding the relationship between the magnetic field and the conductivity beneath the surface of the earth is important for inversion of MMR data. One may expect that different conductivity distributions will give rise to different magnetic fields and hence any change of the conductivity will be manifested in the observed data. This is true for the electric fields, and it allows us to recover absolute values of the conductivity from observed DC resistivity measurements, albeit there is usual non-uniqueness associated with the inverse problem. The situation from MMR data is more complicated. We have already discussed one form of insensitivity, that is, the anomalous surface magnetic field is insensitive to 1-D variations in conductivity. In addition, identical magnetic fields arise from any two conductivities that differ only by a constant scaling factor.

To see this, consider a general conductivity model $\sigma\ (x, y, z)$ and compute potentials $\mathbf{A}$ from eq. (7) and ϕ from eq. (3). These two equations are uncoupled so ϕ can be determined solely from the later equation. Now consider another conductivity model such that $\sigma_2(x, y, z) = \chi\sigma_1(x, y, z)$, where χ is a constant. From eq. (3), it is easy to see that $\phi_2 = \phi_1 / \chi$, i.e., $\sigma_1\nabla\phi_1 = \sigma_2\nabla\phi_2$. So the source term $\mathbf{J}^s - \sigma\nabla\phi$ on the right hand side of eq. (7) remains unchanged for these two models. Since the left hand side does not involve conductivity, the vector potential $\mathbf{A}$, and hence the magnetic field $\mathbf{B}$, are unchanged by this scaling. The same conclusion can also be obtained by working with the discretized eq. (11).

This clearly indicates that the magnetic field is not sensitive to the absolute conductivity but is only dependent upon a ratio of conductivities $\sigma(x,y,z)/\sigma_0$, where σ_0 is a constant background. The two analytic results presented earlier substantiate this. The analytical expressions of the anomalous magnetic field B_z^a are only related to conductivities through the reflection coefficients K_{21} for the vertical contact and C_n for the semispherical model. K_{21} and C_n are expressed as

$$K_{21} = \frac{1-\sigma_2/\sigma_1}{1+\sigma_2/\sigma_1} \quad , \tag{19}$$

and

$$C_n = \frac{1-\sigma_1/\sigma_0}{(n+1)+n\cdot\sigma_1/\sigma_0}, \tag{20}$$

where n is the degree of the Legendre polynomial. It is evident to see that both K_{21} and C_n indeed are only sensitive to the ratio of conductivities, not the absolute conductivity values.

The result that scaled conductivities generate the same magnetic field holds for all measurements irrespective of whether they are acquired inside or outside the earth. In addition, for surface MMR data, there is the complication that the magnetic fields from a 1-D earth are the same as those arising from a halfspace. This does not mean that the anomalous signal from a buried conductor is independent of the background. In particular, sufficiently conductive overburden can completely shield the body.

Nevertheless, we expect to get poor understanding about the vertical variation of conductivity from only surface measurements. To illustrate this, consider the anomalous surface field B_y due to a cube buried in different 1-D conductivities. The cross-section of the 3-D model is shown in Fig. 7d. The conductivities of the first layer and the 3-D cube are unchanged for three models, but the conductivity of the second layer varies from 0.001 S/m to 0.1 S/m. The corresponding anomalous B_y components at the surface are displayed in Figs. 7(a-c). As conductivity of the second layer increases, the amplitude of the B_y decreases significantly from about 80 pT to 12 pT. Two factors contribute this change. As mentioned earlier, the magnetic field is related to the ratio of the conductivities. In this example, the ratio σ_b / σ_2 decreases from 100 to 1, certainly causing a major reduction of the B_y field. At the same time, the magnetic field is also proportional to the amount of anomalous current flowing in the 3-D body, and this is controlled by the 1-D background conductivity. In addition, we see that the three B_y responses have identical shapes, quite similar to a dipole field, and the major difference is only the amplitude. This suggests that it is unlikely to distinguish a 3-D body in a layered earth or in a uniform halfspace if only the surface MMR data are used.

3 3-D Inversion of MMR data

As shown in the previous section, the magnetic field $\mathbf{B}^g$, or $\mathbf{B}^a$ are nonlinearly related to conductivity σ. The goal of the inverse problem is to find a conductivity distribution that can reproduce the observed data to a desired degree. It is well-known in geophysics that this kind of inverse problem is not unique. There are generally infinitely many conductivity models that can fit the observed data equally well. To find a particular model, we can cast the inverse problem as an optimization problem where an objective function of the model is minimized subject to a constraint that the misfit between the observed and the predicted data is in a desired value. We will briefly outline this approach below.

For MMR inversion, the first question that arises concerns the definition of the "model" parameter. As we know, conductivity can vary by orders of magnitude and needs to be positive. A commonly used model parameter in the EM community is to define $m = \ln\sigma$. This choice of model parameter also makes it compatible with DC inversions. Ultimately we would like to pursue a joint MMR-DC inversion in the near future.

Having defined a model, we next construct a model objective function ϕ_m. Our choice for ϕ_m is guided by the fact that we often wish to find a model that has minimum structure in the three spatial directions, and at same time close to the reference model m_0. An objective function that has the flexibility to accomplish these goals is

$$\phi_m(m,m_0) = \alpha_s \int_v [m(\mathbf{r}) - m_0]^2 dv + \alpha_x \int_v \left\{ \frac{\partial [m(\mathbf{r}) - m_0]}{\partial x} \right\}^2 dv$$
$$+ \alpha_y \int_v \left\{ \frac{\partial [m(\mathbf{r}) - m_0]}{\partial y} \right\}^2 dv + \alpha_z \int_v \left\{ \frac{\partial [m(\mathbf{r}) - m_0]}{\partial z} \right\}^2 dv \quad , \qquad (21)$$

where $\alpha_s, \alpha_x, \alpha_y,$ and α_z are coefficients that affect the relative importance of different components in the function.

The form of objective function is particularly appropriate for MMR investigation. With the choice of $m = \ln\sigma$ then $m - m_0 = \ln(\sigma / \sigma_0)$. If σ_0 is a constant, then σ / σ_0 reflects the information that is available from the MMR data alone, namely that the MMR data are not sensitive to absolute conductivity but are indeterminate by a constant factor. In the absence of a prior information we can obtain information only about σ / σ_0. The final conductivity obtained from inversion will be "floating" on this constant reference conductivity. If σ_0 happens to be the conductivity of the real background geology, the inverted $\sigma(x, y, z)$ might be a good approximation to the conductivity distribution of the true geological model. Otherwise, $\sigma(x, y, z)$ can only reveal the relative conductivity variations with respect to the reference model provided.

To perform a numerical implementation, we discretize the model objective function in eq. (21) using a finite-difference approximation according to the mesh defining the conductivity model. The discrete form of the eq. (21) is

$$\phi_m(\mathbf{m}) = \left\| \mathbf{W}_m (\mathbf{m} - \mathbf{m}_0) \right\|^2 \quad . \qquad (22)$$

The derivation of the matrix $\mathbf{W}_m$, which has incorporated the smallest and three derivative components, can be found in Li & Oldenburg (1996).

The next step in setting up the inversion is to define a data misfit between the observed and predicted data. With the same model parameterization, the forward modelling operator is assumed to be written as $\mathbf{d} = F[\mathbf{m}]$, and we can use the 2-norm measure as the data misfit

$$\phi_d = \left\| \mathbf{W}_d (\mathbf{d} - \mathbf{d}^{obs}) \right\|^2 \quad , \qquad (23)$$

where $\mathbf{W}_d$ is a diagonal matrix. If the noise contaminated in the *i*th observation is an uncorrelated Gaussian random variable having zero mean and standard deviation ε_i, then appropriate form for $\mathbf{W}_d$ is $\mathbf{W}_d = diag\{1/\varepsilon_1,\cdots,1/\varepsilon_N\}$, where N is the number of observations. This assumption makes ϕ_d a random variable distributed as chi-square with N degrees of freedom, and thus the expected value of ϕ_d is approximately equal to N, assuming the errors are correctly estimated. Therefore, our target misfit ϕ_d^* for the model sought from the inversion should be set around this value.

The inverse problem is now formulated as the optimization problem: minimize an objective function

$$\phi(\mathbf{m}) = \phi_d + \beta\phi_m(\mathbf{m}) \quad , \tag{24}$$

where β is a regularization parameter.

This problem is nonlinear and iteration is required. Using a standard Gauss-Newton approach so that at each iteration we solve for the model perturbation $\delta\mathbf{m}$

$$\left(\mathbf{J}^T\mathbf{W}_d^T\mathbf{W}_d\mathbf{J} + \beta\mathbf{W}_m^T\mathbf{W}_m\right)\delta\mathbf{m} = -\mathbf{J}^T\mathbf{W}_d^T\mathbf{W}_d\{F[\mathbf{m}^{(n)}] - \mathbf{d}^{obs}\} - \beta\mathbf{W}_m^T\mathbf{W}_m(\mathbf{m}^{(n)} - \mathbf{m}_0) \ , \tag{25}$$

where $\mathbf{J}$ is the sensitivity matrix (or Jacobian matrix) of $N \times M$, having elements

$$J_{ij} = \frac{\partial d_i}{\partial m_j} \ . \tag{26}$$

Calculation of $\mathbf{J}$ and the product of $\mathbf{J}$ or its transpose with a vector will be postponed to Appendix. The choice of regularization parameter β has been addressed by many researchers. We adopt a simple cooling procedure. That has two steps. Firstly, we choose an initial β_0 that is sufficiently large so that $\beta_0\mathbf{W}_m^T\mathbf{W}_m$ dominates $\mathbf{J}^T\mathbf{J}$ component in eq. (25) and hence the problem is nearly quadratic. Secondly, we reduce β as the iterations proceed. In order to determine β_{k+1} from the *kth* iteration, an empirical formula can be used

$$\beta_{k+1} = \lambda\beta_k \ , \tag{27}$$

where λ is a constant, say 0.5. Usually we do not know if this β_{k+1} is appropriate or not. A convergence check (Haber and Oldenburg, 2000) is used. This is achieved by computing the consistency function ϕ^{k+1} at $(k+1)$*th* iteration, which is

$$\phi^{k+1} = \phi_d^{k+1} + \beta_{k+1}\phi_m^{k+1} \ , \tag{28}$$

and then checking if we have decreased the objective function compared to the previous iteration, i.e.

$$\phi^{k+1} < \phi_d^k + \beta_{k+1}\phi_m^k \quad . \tag{29}$$

If this inequality is satisfied, the model will be updated and β_{k+1} is decreased by eq. (27), and move to next iteration; otherwise, the reduction in β_{k+1} has been too large and increase λ in eq. (27), resolve eq. (25), and re-check eq. (29). This process is repeated until the target misfit is achieved.

Now we need to address how to solve the matrix equation in eq. (26) for the perturbation $\delta\mathbf{m}$ at each iteration. Solving eq. (26) is identical to solving the equivalent least squares problem

$$\begin{pmatrix} \mathbf{W}_d\mathbf{J} \\ \sqrt{\beta}\mathbf{W}_m \end{pmatrix}\delta\mathbf{m} = \begin{pmatrix} -\mathbf{W}_d(F[\mathbf{m}^{(n)}] - \mathbf{d}^{obs}) \\ -\sqrt{\beta}\mathbf{W}_m(\mathbf{m}^{(n)} - \mathbf{m}_0) \end{pmatrix}. \tag{30}$$

We use the conjugate gradient least-squares algorithm (CGLS) (Golub & van Loan, 1996) to solve eq. (30). The main computations required for this algorithm are the product of the matrix $\mathbf{J}$ with a vector and the product of the its transpose matrix $\mathbf{J}^T$ with a vector. The derivation is carefully explained in Appendix.

4 Practical considerations of the inversion

4.1 Depth weighting

As the first example of synthetic data study, we invert the single component MMR anomaly data (y-component) as defined in the previous section. The model consists of a 3-D cube buried in a uniform half-space. The conductivities of the cube and half-space are set to be 0.1 S/m and 0.001 S/m, respectively. For simplicity, the cube is located right below the origin of the coordinates, with a top depth of 80 m and side length of 400 m. Fig. 8a shows the cross-section of the true model at y = 0 plane. The source and sink electrodes are located at –600 m and 600 m along the x-axis. The measurement area at the surface extends from –400 m to 400 m in both directions, with 25 survey lines and 25 sites on each line, resulting in a total number of 625 data points. The cube produces the MMR anomalies which are shown in Fig. 8c. The MMR anomaly is obtained by normalizing the anomaly magnetic field (y-component here) by the normal field B^n at the center of the survey area, defining the so-called "MMR anomaly" in percentage

$$\text{MMR anomaly} = 100 \times \frac{B^a}{B^n} \quad . \tag{31}$$

The data have independent Gaussian noise added whose standard deviation is equal to 5% of the accurate datum plus a constant error of 0.5. We invert these 625 noise-contaminated data to recover the conductivity of an earth model parameterized by 32x32x15 cells (the air layer is excluded). The reference model is the same as the true uniform half-space of 0.001 S/m. After 7 iterations, the final misfit is 586 and the cross-section through the center of the recovered model is shown in Fig. 8b. The predicted MMR data are also shown in Fig. 8d for comparison with Fig. 8c.

The inverted conductivity model tends to be concentrated near the surface of the earth. The contrast of conductivity between the target and the host (0.6 in a log scale) is also much smaller than the true value of 2 (or 100:1 in a linear scale). These results may have been anticipated because of the close relationship between the magnetic field due to a volume of current distribution and the gravity field due to a volume of density variation. The magnetic field **B** is obtained by the Biot-Savart law which is given by

$$\mathbf{B} = \frac{\mu}{4\pi} \int_v \mathbf{J} \times \nabla(\frac{1}{r}) dv \quad , \tag{32}$$

while the gravity field $\mathbf{g}$ is given by

$$\mathbf{g} = G \int_v \rho \nabla(\frac{1}{r}) dv \quad , \tag{33}$$

where G is the universal gravitational constant and ρ is the density. Both have the same kernel function $\nabla(\frac{1}{r})$. When expanding eq. (32), it is easy to find that each component of **B** can be expressed as a linear combination of two components of gravity field. For example, B_y component can be written as $B_y = g_x - g_z$, provided that the density ρ is replaced with $\frac{\mu}{4\pi G} J_z$ when computing g_x and with $\frac{\mu}{4\pi G} J_x$ when computing g_z. For a 2-D structure, as first recognized by Szarka (1987), $B_y = -g_z$ if the current flows only parallel to the strike direction (x-direction here). Thus the magnetic field can be estimated by existing gravity formulae (Asten, 1988).

It is well known that gravity data have no inherent depth resolution. To counteract the geometric decay of the kernels and to distribute the density with depth, Li & Oldenburg (1998) introduce a depth weighting into the model objective function. The established similarity between the magnetic field and gravity allows us use the same depth weighting for inversion of MMR data. The weighting is incorporated into the inversion, by altering the model objective function given by

$$\begin{aligned}\phi_m(m,m_0) = \alpha_s \int_v \{w(z)[m(\mathbf{r})-m_0]\}^2 dv + \alpha_x \int_v \left\{\frac{\partial w(z)[m(\mathbf{r})-m_0]}{\partial x}\right\}^2 dv \\ + \alpha_y \int_v \left\{\frac{\partial w(z)[m(\mathbf{r})-m_0]}{\partial y}\right\}^2 dv + \alpha_z \int_v \left\{\frac{\partial w(z)[m(\mathbf{r})-m_0]}{\partial z}\right\}^2 dv\end{aligned}, \qquad (34)$$

where $w(z)$ is the depth weighting function. This is discretized in the same manner as eq. (21).

For surface MMR survey, the depth weighting $w(z)$, similar to the one implemented in Li & Oldenburg (1996), takes the form of $w(z) = 2z_0/(z+z_0)^\gamma$. The parameters z_0 and γ are chosen so that $w^2(z)$ is approximately equal to the decay of the kernels, so $\gamma \approx 1$ emulates the $1/r^2$ decay of gravity kernels. z_0 depends upon cell size and observation height (if there is topography). Here we choose z_0 to be half the thickness of the cell just below the surface, z is the depth to a cell center, and $\gamma = 0.95$.

We have tested the weighting functions by inverting the noise-contaminated data from the buried cube in the first example. z_0 is equal to 10 m. The reference model is the uniform half-space of 0.001 S/m. Fig. 9 shows the cross-section through the center of the inverted conductivity model, which can be compared with Fig. 8a. The inverted model with the depth weighting is shifted to depth and approximately coincides with the true model. The outline of the constructed model is in reasonable agreement with the true model, and the contrast of the conductivities is increased to about 1.5 (in log scale), much closer to the true contrast of 2.

4.2 Issues related to reference model

The choice of reference model is another key element in our inversion algorithm. As explained in the previous section, the MMR anomaly is only sensitive to the relative conductivity contrast between targets and their surroundings. Therefore, the inverted model must be interpreted in conjunction with the reference model. In order to reveal the true conductivity structures, the reference model should be as close to the true background as possible. Without this knowledge of background to calibrate the models, however, the MMR inversion still can provide useful information about the relative conductivity contrast. We use the following example to illustrate this point.

We return to the cube model with all inversion parameters being exactly the same as those used in the depth weighting experiment. The only difference is the choice of the reference model. Two reference models are tested. One is a uniform half-space of 0.01 S/m; the other 0.1 S/m. The cross-sections of the inverted models through the center of the true cube model are shown in Fig. 10a and 10b, respectively. For comparison, these results can be judged in conjunction with the true model and the inverted model using 0.001 S/m reference model, both of those are shown in Fig. 8a and 9. Clearly, the shapes of the inverted anomalies are almost identical for the three different models. They all match quite well with the outlines of the true cube. Although the conductivity values shown in the gray bar are changed with the reference models, the contrast (1.5 in a log scale) is almost maintained. This verifies that the choice of constant for a reference model does not affect the recovery of the conductivity contrast structure in our inversion.

The effect of a conductive overburden on the MMR inversion is closely related to the choice of reference model. Suppose that there is a moderately conductive overburden overlying a basement in which a target resides. Surface measurements are acquired with the goal of recovering structures below the overburden. There are two practical issues to be addressed. First, can a significant MMR response from the target be obtained in the presence of the conductive overburden. In the conventional DC survey, the "masking effect" has severely limited DC's application in areas where an overburden or weathered layer has conductivity one or more orders of magnitude higher than the conductivity of the underlying rock. The second aspect addresses the question of whether the relative conductivity structure of the target can be recovered even though a uniform half-space is specified as the reference model.

To get some insight about the first issue, we take the cube model as an example, but add an overburden with conductivity of 0.01 S/m and thickness of 30 m. Other geometric and electrical parameters remain unchanged. Figs 11a and 11b show the y-component of the anomalous magnetic field for the models without, and with, the conductive overburden, respectively. The maximum magnitude of the magnetic field decreases from 72 pT to 65 pT, a change of only about 10%. This indicates that we may "see" the target of interest through a moderately conductive cover layer, and this constitutes a principal advantage of MMR survey against the direct current resistivity method. The signal strength for the DC potentials decreases by 50% in this same example.

To address the second question, we invert the MMR data produced from the cube model with the conductive overburden. The data are first contaminated with random Gassian noise of 5%, the depth weighting is imposed, and the conductivity of the reference model is 0.001 S/m. A cross-section of the recovered structure in Fig. 12b, shows that the location of the target and conductivity contrast with the background are reasonably well defined. The inversion provides no indication of the 1-D overburden structure. This is a practical consequence of the fact that surface data from any 1-D conductivity structure is the same as that of a uniform half-space. The reconstruction of the target body however has been good because the conductive overburden has not greatly changed the amount and distribution of current going through the prism. When the overburden is thicker, or more conductive, its effect will become more pronounced. In such cases, the choice of reference model becomes more critical. Quantifying these effects is left for future research.

4.3 Which component data to invert ?

In the previous synthetic examples, we inverted only the y-component of the anomalous MMR response. The choice was tied to practical signal-to-noise issues. With the typical acquisition of MMR data using a U-shaped current layout for the wires, then the y-component of the anomalous magnetic field usually has the highest fidelity. This can be understood as follows.

As explained in the forward modeling section, the total magnetic fields consist of three parts: $\mathbf{B}^w, \mathbf{B}^n$ and $\mathbf{B}^a$. The magnetic field $\mathbf{B}^w$ due to current flow in the wire can be directly obtained through the Biot-Savart law. For the case of our cube model, the U-shaped wire has three segments, each of those is 1200 m long, and is assumed to be laid on the side of y<0. Because the measurements are made at the surface, $\mathbf{B}^w$ only has vertical component. The other two horizontal components are null. Deviations of wire into the vertical direction will generate horizontal components. It is for this reason that the U-shape was adopted so that the wire is away from the location at which fields are to be measured. In contrast, the normal magnetic field $\mathbf{B}^n$ has no vertical component, but just two horizontal ones, which can be calculated analytically. The anomalous magnetic fields $\mathbf{B}^a$ have been shown in Fig. 1. Adding these three parts together will form the total magnetic fields which are shown in Fig. 13. This Fig. clearly shows that B_z has the largest amplitude of about 1200 pT; the second is B_y, 650 pT, and the least is B_x, 200 pT. From the viewpoint of measurements, B_z might be easiest to record. However, only the anomalous fields contain useful information about the target of interest and so we invert them. From Fig. 1, the amplitudes for the vertical and y-component of the anomalous fields are about the same size, 60 pT and 70 pT, respectively; the x-component is roughly one-third of the y-component. Thus, B_z and B_y might be two alternatives. In practice, the layout of the wire along the three sides may be not straight, or the surface of the earth is not a flat as it is supposed to be. Both situations will make B_z more noise-prone compared to B_y and hence it is usually only that component which is collected and inverted. In principle however, if the location of the current wire is accurately recorded then its effect can be subtracted and all components contain valuable information about the subsurface conductivity variations.

Fig. 14 is the cross-section of the recovered conductivity model obtained by inverting only the vertical component data. Five percent of noise was added to the data prior to inversion. It compares favorably with Fig. 12b, which was obtained from inverting the y-component. Both the location of the cube and the conductivity contrast are reasonably well defined.

5 Field example

Surface MMR data were collected at the Mons Cupri deposit of the Pilbara area, Western Australia. As shown in Fig. 15, the general strike of the formation around the Mons Cupri deposit is north-south and dips 30 degrees westward. The ore body is hosted by the Mons Cupri rhyolite fragmental, which is sequentially overlain by the Cistern Formation, Cap shale, Comstock andesite and Whim Creek shale. Mineralization consists of a zone of iron-rich chlorite and carbonate alteration containing disseminated and stockwork chalcopyrite mineralization overlain by a shallowly dipping massive sphalerite galena, chalcopyrite lens in volcaniclastic and cherty sediments (Linford, 1990). The stockwork zone, containing copper mineralization in chalcedonic silica and carbonate veins, dips steeply to the south, and is approximately 1 km long, extending to 200 m below the present surface. The uppermost part of the ore body is oxidized down to 100 m below the surface. It has been estimated that the deposit's resource is 1.5 Mt oxide ore at 1.13 wt% Cu and 1.4 Mt sulfide ore at 1.74 wt% Cu, 1.13 wt% Pb, and 2.48 wt% Zn.

MMR surveys, together with MIP, were conducted over the deposit by Scintrex (Linford, 1990). The current electrodes were set on 1000N at 1500E and 300E, and are aligned with the geologic cross-section. An area of 1000 m by 1000 m over the ore body was surveyed. Eleven survey lines parallel to the strike were 100 m apart, and the station interval inside the central 500 m by 500 m area was 50 m and it was 100 m outside this central area. Data were acquired at 0.3 Hz. The observed horizontal magnetic field (South pointing) was reduced to the MMR response in percent by the procedure discussed earlier. Fig. 16a shows the MMR responses at the 162 stations. Basically, there are two larger MMR highs which are in red. One, at line 1400N, extends from 500E to 800E, and has a magnitude of about 104%. From Fig. 15, this anomaly covers the marked Cu gossan and mineralization at the upper left corner. The other MMR high is much smaller and is located at about 950N, and extends from 1100E to 1250E. Interestingly, this anomaly is directly above the known massive sulphides. These two highs suggest that the mineralization is more conductive compared with its surroundings. Other local highs may be not trustworthy because only a single data point is involved. There is a large negative anomaly at 1000N of the West side of survey area. The anomaly is as low as –162%, indicating bodies that are more resistive than the host medium.

The survey area is characterized by a flat terrain except for the small scattered hills resistant to the erosion, so a 3-D model of 2 x 2 x 1 km^3 (excluding the air space), without topography, was designed. The model was discretized horizontally at a non-uniform interval, from 50 m in the central 1000 m by 1000 m area, to 100 m outside. In the vertical direction, the first 300 m was divided at a 25 m interval so that the shallow structure could be adequately modeled. Below that an interval of 50m to 100 m was used. This resulted in a mesh with 34 x 34 x 20 cells. The inverse problem was therefore formalized by inverting 162 data to recover the conductivities in these 23,120 cells. The reference model was a uniform half-space of 0.001 S/m. Little is known about the conductivity of the different rock units but they are likely resistive. Two core sample measurements exist (Linford, 1990). The resistivity of copper-disseminated sulphide and oxide sample is 3,000 ohm-m, while a copper-massive sulphide sample is less than 1 ohm-m. For the inversion, a depth weighting function with $\gamma = 0.95$ and $z_0 = 12.5\,\mathrm{m}$ was adopted. We also assumed that each datum had an error whose standard deviation is equal to 8% of its magnitude plus a base value of 2 anomaly units. The target misfit was set to 162, but the achieved misfit after 13 iterations was 850. The predicted data are shown in Fig 16b.

The recovered conductivity model is shown in Fig. 17 as one plan-section, marked by the sketch of the two known deposits, at depth of 85 m. The maximum recovered conductivity value is about 58 mS/m, much smaller than the core sample conductivity of 1 S/m for the massive sulphides. This may be explained in two ways. Firstly, the saturation effect of MMR responses prevents the algorithm producing a target whose conductivity value is 100 times greater than its surrounding's. In this case, the reference model has conductivity of 1 mS/m. Therefore, the conductivity for the target should be less than 100 mS/m. Secondly, the MMR response on the surface is produced by a bulk volume with certain size underground. The recovered "bulk" conductivity is likely less than the "point" core sample value. There are two other regions of high conductivity. One is in the upper right corner, and the other is at 700N, below the massive deposit. The blue represents high resistive materials (0.1 mS/m), which accounts for the negative MMR response on the left side of the central deposit. These high resistive materials may be associated with Whim Creek shale and some volcanics.

Fig. 18 compares the recovered conductivity model with the geology in the cross-section at 950N. The Lead-Zinc mineralization consists of two tabular targets separated by about 200 m in depth. The MMR results show a region of high conductivity centered between the two mineralized lenses. This is characteristic of a low resolution image of a complex structure. Overall, we feel that the inversion has been successful in delineating the volume containing the mineralization, but the resolution is poor. Acquisition of other components of the magnetic field, and especially, obtaining data from one or more other locations of the current electrodes, could greatly improve the results.

6 Discussions

We have developed an algorithm to compute the MMR response due to steady current sources in a 3-D environment. This is achieved in two consecutive steps: first solving a DC resistivity problem and then solving a magnetostatic subproblem. The differential equations involved in these two problems are numerically solved by using the finite-volume technique based on a staggered grid. The algorithm is versatile: the grounded sources can be located anywhere although we assumed the sources were at the surface of the earth in our examples; topography can be included into the model; and highly discontinuous conductivity and susceptibility can also be handled. The algorithm has been tested against analytical models such as a vertical contact and hemispherical depression.

We also have developed a technique to invert the surface MMR data to recover a 3-D conductivity distribution. To overcome the inherent nonuniqueness of the inverse problem, we obtain the solution by minimizing a specific model objective function subject to a target misfit. Because the static magnetic field has no depth resolution, a depth weighting function has to be included into the objective function. A crucial element in solving a large-scale inverse problem is that we avoid computing and storing the sensitivity matrix explicitly; instead we only need to calculate the sensitivity matrix and its transpose multiplying a random vector, which can be accomplished with two extra forward modellings. The minimization is carried out using the Gauss-Newton method in which the perturbation at each iteration is obtained by solving an equivalent conjugate gradient least squares problem. The regularization parameter is pragmatically determined in a cooling process.

Through this research work, we have shown that there is a fundamental non-uniqueness in that conductivity is always ambiguous by a multiplicative constant. For a body buried in a uniform host medium, we can determine only the relative conductivity contrast. We need, in principle, one additional piece of information, such as a conductivity somewhere, to calibrate the result. From the observation that MMR response is only sensitive to the ratio of conductivity between the target and the surrounding, we have chosen a constant conductivity as reference model. Obtaining information about the relative conductivity contrast is facilitated by the form of our objective function and using a constant conductivity as a reference model. The choice of a constant reference model does not have any effect on the recovery of the relative conductivity structure. In addition, when receivers are at the surface there is an additional ambiguity because the field from 1-D earth is the same as that of a uniform halfspace. In practice this means we are insensitive to 1-D variations in conductivity, and it is unlikely to obtain information about then 1-D conductivity from surface MMR surveys only.

A more complicated scenario is a 3-D body residing in a 1-D layered medium. The anomalous signal we measure is not only related to the ratio of conductivity between the 3-D body and its surroundings, but also controlled by the conductivity structure of the 1-D earth. This is because the measured signal is proportional to the amount of anomalous current in the body. When the host medium does not significantly change the current distribution in the earth, such as a 3-D body with a thin and moderately conductive overburden, the signal will not be altered much compared with that in a uniform host, and thus it is possible to reconstruct the relative conductivity of the 3-D body. Otherwise, the responses will be altered significantly. Quantifying this interaction effect between host and target on the inversion result is a subject of future research.

The work provided here is general. We show how to invert magnetic data that arise from any steady state current. We have presented examples that are associated with a traditional MMR geometry used in mineral exploration. The field example was deemed to be successful in that the low resolution image from the inversion seemed to correspond with the major geological units (in particular the mineralization), however it also highlights the deficiencies in the traditional field approach. Only one source location and one component of magnetic field data were used in the inversion. More source locations and acquisition of full 3-component data would greatly improve the results. We hope that the ability to invert such data will prompt these improved survey techniques.

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Figure captions

Figure 1. Anomalous magnetic fields at the surface produced by a 400x400x400 m^3 cube (d) with conductivity of 0.1 S/m and a top depth of 80m, embedded in a host of 0.001 S/m. The source and sink electrodes are located at the x-axis, 600 m each from the origin of the coordinates. (a) x-component; (b) y-component; (c) z-component. The magnitudes are in pecoTesla (pT).

Figure 2. A schematic geometry of the surface MMR survey**.**

Figure 3. A grid cell showing the locations for each unknown variable. ϕ and ψ are at the center of the cell; A and J are at the center points of the faces, denoted by arrows with dashed tails; H is at the edges**.**

Figure 4. **D**iagram defining the angles and unit vectors used to determine the magnetic field $\mathbf{B}^w$ due to current flow of finite length straight wire segment.

Figure 5. 3-D numerical vertical component of anomalous magnetic field (a) and its percentage error (b), compared with analytical solution for a vertical contact model shown in (c). The source and sink electrodes, carrying 1.0 Ampere, were located at (0, -600, 0) and (0, 600, 0) meters, respectively**.**

Figure 6. 3-D numerical vertical component of anomalous magnetic field (a) and its percentage error (b), compared with analytical solution for a hugely conductive hemi-sphere of radius 200 m shown in (c). The source and sink electrodes, carrying 1.0 Ampere, were located at (0, -600, 0) and (0, 600, 0) meters, respectively**.**

Figure 7. Y-components of the anomalous magnetic field at the surface produced by 3-D models shown in (d). The conductivity of the basement varies: (a) 0.001 S/m ; (b) 0.01 S/m; and (c) 0.1 S/m.

Figure 8. 3-D inversion results of synthetic MMR anomaly (y-component) without depth weighting. The true model is the same as the one described in Fig. 1. (a) Cross-section through the center of the cube; (b) Cross-section of the inverted model. The reference model is a half-space of 0.001 S/m. The gray scales beside both indicates the conductivity (S/m) in a logarithmic scale. (c) The synthetic anomaly produced by the cube model. Uncorrelated Gaussian noise, with a standard deviation of 5% of the datum magnitude plus 0.5, is added to form the data; (d) the predicted data produced by the inverted model. The gray scales on the bottom two plots show the MMR anomaly in percentage.

Figure 9. Cross-sections through the center of the inverted model by using the y-component data. The true model is shown in Fig. 8a. The inversion uses the depth weighting function discussed in the text. The reference model is a half-space of 0.001 S/m**.**

Figure 10. Cross-sections through the center of the inverted models with a reference model of 0.01 S/m (a) and 0.1 S/m (b). Note the inverted conductivity values, denoted in logarithm, change with the reference models, but the relative conductivity contrast is principally unchanged.

Figure 11. Synthetic y-components of anomalous magnetic field for the cube model: (a) without, and (b) with, a 30 m thick conductive overburden of 0.01 S/m. The magnitudes of the magnetic field are in pT. The conductive overburden reduces the anomalous field only by 10%.

Figure 12. Cross-sections through the center of the true model with a conductive overburden (a) and of the inverted model (b) obtained by inverting the y-component data. The depth weighting function is applied, and the reference model is a half-space of 0.001 S/m.

Figure 13. The total magnetic fields produced by the cube model, which are composed by three parts: $\mathbf{B}^w, \mathbf{B}^n$, and $\mathbf{B}^a$. $\mathbf{B}^a$ is shown in Fig. 1.

Figure 14. Cross-sections through the center of the inverted model by inverting the z-component data. The true model is shown in Fig. 12a. The depth weighting function is applied, and the reference model is a half-space of 0.001 S/m**.**

Figure 15. Geological plan and section view of the Mons Cupri deposit with current electrodes, connecting wires and the MMR survey area**.**

Figure 16. (a) The MMR response (%) over the Mons Cupri deposit. The survey area is 1000 m x 1000 m, with a line spacing of 100 m and a station interval of 50 m in the central area and 100 m outside, resulting in 162 stations (marked by dots). See Figure 15 for reference. The measured data are horizontal (South) components of magnetic field at frequency of 0.3 Hz. (b) The predicted response from the recovered model**.**

Figure 17. The recovered conductivity model shown in plan-section at depth of 85 m. The known Cu gossan and mineralization at the surface are also marked.

Figure 18. Comparison of the recovered conductivity model in a cross-section (950N) with the geology for the Mons Cupri deposit. The conductivity high is correlated to the deposit, and the resistive basement relates to the Rhyolite Fragmental**.**

Appendix Computation of $\mathbf{J}\cdot\mathbf{v}$ and $\mathbf{J}^T\cdot\mathbf{v}$

For convenience, we use a new notation (more details can be referred to Haber *et al.*, 2000b)

$$\mathbf{A}(\mathbf{m})\cdot\mathbf{u}=\mathbf{f} \quad , \tag{A-1}$$

to simplify the forward modelling matrix systems as given in eq. (11). The coefficient matrix is

$$\mathbf{A}(\mathbf{m})=\begin{pmatrix} \nabla_h^{(e)}\times\mathbf{M}_e^{-1}\nabla_h^{(f)}\times \;-\; \nabla_h\mathbf{M}_c^{-1}\nabla_h\cdot & \mathbf{S}\nabla_h \\ 0 & \nabla_h\cdot\mathbf{S}\nabla_h \end{pmatrix} \quad , \tag{A-2}$$

the unknown vector $\mathbf{u}=\begin{pmatrix}\widetilde{\mathbf{A}}\\ \phi\end{pmatrix}$, and the source vector $\mathbf{f}=\begin{pmatrix}\mathbf{J}^s\\ \nabla_h\cdot\mathbf{J}^s\end{pmatrix}$, where $\widetilde{\mathbf{A}}$ is the vector potential. By doing so, the data vector $\mathbf{d}$ can be written as

$$\mathbf{d}=\mathbf{Q}\mathbf{u} \quad , \tag{A-3}$$

where $\mathbf{Q}$ is a projection matrix which generates the data from the computed fields. If the $\mathbf{B}$ field is required, $\mathbf{Q}$ can be obtained by multiplying a linear interpolation matrix with the curl matrix $\nabla_h^{(f)}\times$. Therefore, $\mathbf{Q}$ is independent to the model $\mathbf{m}$.

To explicitly express the sensitivity matrix $\mathbf{J}$ in the new notation, we differentiate the discretized differential eq. (A-1) with respect to $\mathbf{m}$ (assuming $\mathbf{u}$ is a function of $\mathbf{m}$). This will yield

$$\frac{\partial[\mathbf{A}(\mathbf{m})\mathbf{u}(\mathbf{m})]}{\partial\mathbf{m}}=\frac{\partial[\mathbf{A}(\mathbf{m})\mathbf{u}]}{\partial\mathbf{m}}+\mathbf{A}(\mathbf{m})\frac{\partial\mathbf{u}(\mathbf{m})}{\partial\mathbf{m}}=0 \quad . \tag{A-4}$$

We take a new matrix $\mathbf{G}(\mathbf{m},\mathbf{u})$ to represent the first term, i.e.,

$$\mathbf{G}(\mathbf{m},\mathbf{u})=\frac{\partial[\mathbf{A}(\mathbf{m})\mathbf{u}]}{\partial\mathbf{m}} \quad , \tag{A-5}$$

where **u** is just a vector and not related to **m**. Substituting $\mathbf{G}(\mathbf{m},\mathbf{u})$ into eq. (A-4) will lead to

$$\frac{\partial \mathbf{u}(\mathbf{m})}{\partial \mathbf{m}} = -\mathbf{A}^{-1}(\mathbf{m})\mathbf{G}(\mathbf{m},\mathbf{u}) \ . \tag{A-6}$$

Thus, the sensitivity matrix $\mathbf{J}$ can be symbolically written as

$$\mathbf{J} = \frac{\partial \mathbf{d}}{\partial \mathbf{m}} = \frac{\partial [\mathbf{Q}\mathbf{u}(\mathbf{m})]}{\partial \mathbf{m}} = \mathbf{Q}\frac{\partial \mathbf{u}(\mathbf{m})}{\partial \mathbf{m}} = -\mathbf{Q}\mathbf{A}^{-1}(\mathbf{m})\mathbf{G}(\mathbf{m},\mathbf{u}) \, . \tag{A-7}$$

The intermediate matrix $\mathbf{G}(\mathbf{m},\mathbf{u})$ can be derived as follows. From eq. (A-2) we see that only material property matrix $\mathbf{S}$ is related to the model $\mathbf{m}$, therefore

$$\mathbf{G}(\mathbf{m},\mathbf{u}) = \frac{\partial [\mathbf{A}(\mathbf{m})\mathbf{u}]}{\partial \mathbf{m}} = \begin{pmatrix} \dfrac{\partial (\mathbf{S}\nabla_h \phi)}{\partial \mathbf{m}} \\ \nabla_h \cdot \dfrac{\partial (\mathbf{S}\nabla_h \phi)}{\partial \mathbf{m}} \end{pmatrix} \ . \tag{A-8}$$

Because $\mathbf{S}$ is a diagonal matrix with elements which are the harmonic averaging of conductivities at adjacent two cells, $\dfrac{\partial (\mathbf{S}\nabla_h \phi)}{\partial \mathbf{m}}$ can be analytically obtained without too much work.

The products of the sensitivity matrix and its transpose with a vector are now readily computed. Consider $\mathbf{J}\cdot\mathbf{v}$, where $\mathbf{v}$ is a known vector.

$$\mathbf{J}\cdot\mathbf{v} = -\mathbf{Q}\mathbf{A}^{-1}(\mathbf{m})\mathbf{G}(\mathbf{m},\mathbf{u})\mathbf{v} \quad , \tag{A-9}$$

and we can obtain it in three steps. First compute $\mathbf{G}(\mathbf{m},\mathbf{u})\mathbf{v} = \mathbf{q}$. Then calculate $\mathbf{A}^{-1}(\mathbf{m})\mathbf{q} = \mathbf{w}$, which is equivalent to solving the forward modelling problem $\mathbf{A}\ (\mathbf{m})\mathbf{w} = \mathbf{q}$, addressed in the main body of this paper. Once $\mathbf{w}$ is obtained, the last step is just to multiply $\mathbf{w}$ by $-\mathbf{Q}$, i.e., $\mathbf{J}\cdot\mathbf{v} = -\mathbf{Q}\mathbf{w}$.

Then we turn to deal with $\mathbf{J}^T\mathbf{v}$, which can be expanded as

$$\mathbf{J}^T\cdot\mathbf{v} = -(\mathbf{Q}\mathbf{A}^{-1}(\mathbf{m})\mathbf{G}(\mathbf{m},\mathbf{u}))^T\,\mathbf{v} = -\mathbf{G}^T(\mathbf{m},\mathbf{u})\mathbf{A}^{-T}(\mathbf{m})\mathbf{Q}^T\mathbf{v} \ . \tag{A-10}$$

Similarly, we can define $\mathbf{q} = \mathbf{Q}^T\mathbf{v}$, and $\mathbf{w} = \mathbf{A}^{-T}(\mathbf{m})\mathbf{q}$. This means $\mathbf{A}^T(\mathbf{m})\mathbf{w} = \mathbf{q}$, i.e.,

$$\begin{pmatrix} (\nabla_h^{(e)}\times\mathbf{M}_e^{-1}\nabla_h^{(f)}\times \ - \ \nabla_h\mathbf{M}_c^{-1}\nabla_h\cdot)^T & 0 \\ (\mathbf{S}\nabla_h)^T & (\nabla_h\cdot\mathbf{S}\nabla_h)^T \end{pmatrix}\mathbf{w} = \mathbf{q} \quad . \tag{A-11}$$

As mentioned earlier, eq. (A-11) can be decoupled into two systems. Let $\mathbf{w} = \begin{pmatrix} \mathbf{w}_a \\ \mathbf{w}_\phi \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} \mathbf{q}_a \\ \mathbf{q}_\phi \end{pmatrix}$, then eq. (A-11) can be written as

$$(\nabla_h^{(e)} \times \mathbf{M}_e^{-1} \nabla_h^{(f)} \times \;-\; \nabla_h \mathbf{M}_c^{-1} \nabla_h \cdot)^T \mathbf{w}_a = \mathbf{q}_a \quad , \tag{A-12a}$$

and

$$(\nabla_h \cdot \mathbf{S} \nabla_h)^T \mathbf{w}_\phi = \mathbf{q}_\phi - (\mathbf{S} \nabla_h)^T \mathbf{w}_a \quad . \tag{A-12b}$$

We can first solve for $\mathbf{w}_a$ from eq. (A-12a), substitute $\mathbf{w}_a$ into the right side of eq. (A-12b), and then solve for $\mathbf{w}_\phi$ from (A-12b). Once $\mathbf{w}_a$ and $\mathbf{w}_\phi$ have been computed, $\mathbf{J}^T \mathbf{v}$ can be obtained from

$$\mathbf{J}^T \cdot \mathbf{v} = -\mathbf{G}^T(\mathbf{m}, \mathbf{u}) \mathbf{w} \quad . \tag{A-13}$$

Roughly speaking, computing $\mathbf{J} \cdot \mathbf{v}$ and $\mathbf{J}^T \mathbf{v}$ is equivalent to running the forward modelling twice.