Computational techniques for electromagnetic problems in 3D

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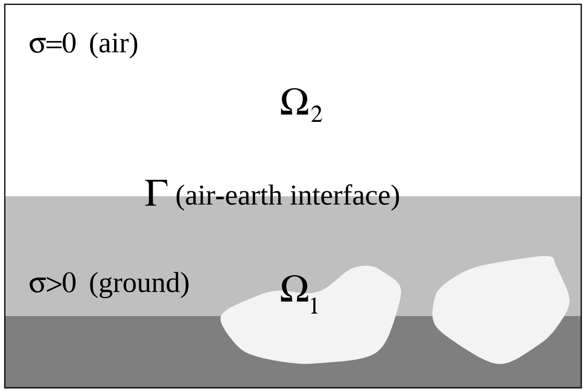
Electromagnetic prospecting

Oil and mining exploration

$$\nabla \times \left(\mu^{-1} \nabla \times \mathbf{E}\right) - \imath\omega\,\sigma\,\mathbf{E} = \imath\omega\mathbf{s}$$



Outline

- The electromagnetic problem
- Discretization and difficulties
- Decomposition of the electric field
- Elements of a finite volume discretization
- Solving the linear algebra equations
- Multigrid preconditioning
- The discrete point of view

The electromagnetic problem

Maxwell's equations in frequency domain

$$
\begin{aligned}
\nabla\times \mathbf{E} - \imath\omega\mu\mathbf{H} &= \mathbf{0}, \\
\nabla\times \mathbf{H} - \widehat{\sigma}\mathbf{E} &= \mathbf{s},
\end{aligned}
$$

where

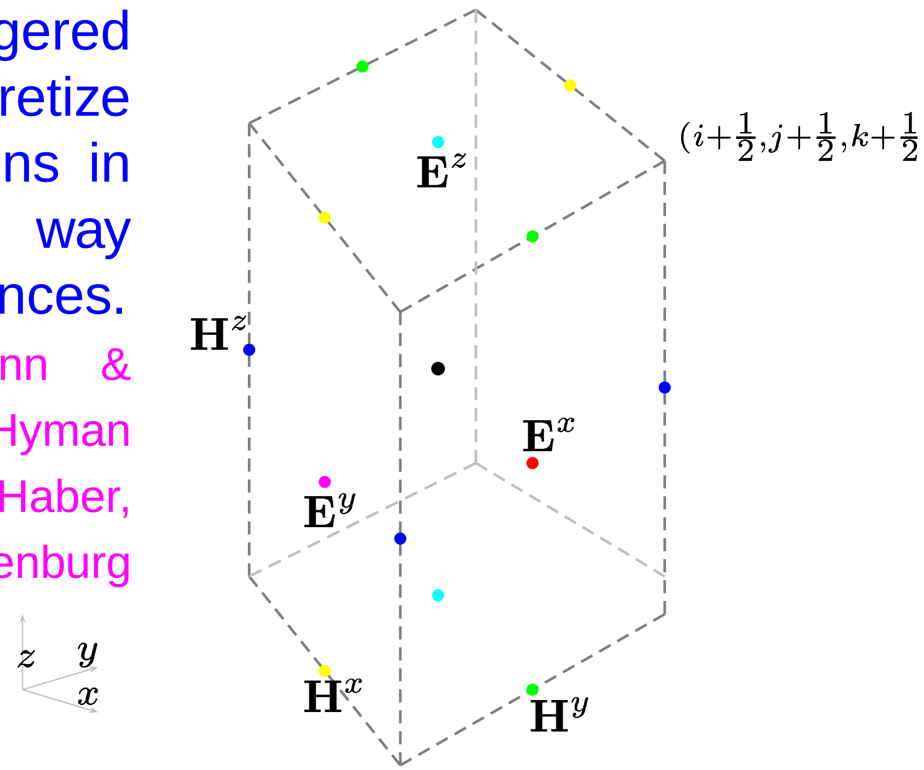
$$\widehat{\sigma} = \sigma - \imath\omega\epsilon \quad = \exp(\mathrm{m})$$

Consider frequencies in low to medium range

$$\nabla\times (\mu^{-1}\nabla\times \mathbf{E}) - \imath\omega\widehat{\sigma}\mathbf{E} \quad = \quad \imath\omega\mathbf{s}$$

Discretization and difficulties

Use a staggered grid. Discretize Maxwell's equations in a straightforward way using short differences. [Yee '66; Newmann & Alumbaugh '95; Hyman & Shashkov '99; Haber, Ascher, Aruliah & Oldenburg '00]



But the resulting large, sparse system of linear equations is difficult to solve efficiently.

This is because $\nabla \times \boldsymbol{\nabla}\psi = 0 \;\; \forall \psi$

Decomposition of the electric field

Helmholtz decomposition + Coulomb gauge :

$$\mathbf{E} = \mathbf{A} + \boldsymbol{\nabla}\phi$$
$$\nabla \cdot \mathbf{A} = 0$$

Thus, $\mathbf{A}$ is the electric field induced by magnetic fluxes, $\boldsymbol{\nabla}\phi$ is due to charge accummulation.

$$\nabla \times \left(\mu^{-1}\nabla \times \mathbf{A}\right) - \imath\omega\widehat{\sigma}(\mathbf{A} + \boldsymbol{\nabla}\phi) = \imath\omega\mathbf{s}$$
$$\nabla \cdot \mathbf{A} = 0$$

Decomposition of the electric field

$$\nabla \times \left(\mu^{-1} \nabla \times \mathbf{A}\right) - \imath\omega\widehat{\sigma}(\mathbf{A} + \boldsymbol{\nabla}\phi) = \imath\omega\mathbf{s}$$

$$\nabla \cdot \mathbf{A} = 0$$

- If $\mu = constant$ then $\nabla \times \nabla \times \mathbf{A} - \boldsymbol{\nabla}\nabla \cdot \mathbf{A} = -\nabla^2\mathbf{A}$. So, stabilize by adding a vanishing term.
- Analogously to the *PPE* in CFD and to index reduction in *DAEs*, differentiate and substitute.

$$\nabla \times\ (\mu^{-1}\nabla \times\ \mathbf{A}) - \boldsymbol{\nabla}(\mu^{-1}\nabla \cdot\ \mathbf{A}) - \imath\omega\widehat{\sigma}(\mathbf{A} + \boldsymbol{\nabla}\phi) = \imath\omega\mathbf{s}$$
$$-\nabla \cdot\ \widehat{\sigma}(\mathbf{A} + \boldsymbol{\nabla}\phi) = \nabla \cdot\ \mathbf{s}$$

For $\widehat{\sigma} = \sigma > 0$ this is an elliptic, diagonally dominant system.

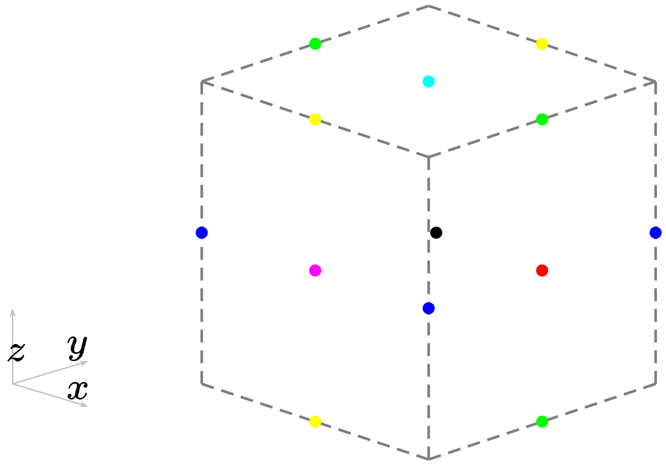
Boundary conditions:

$$\begin{aligned} -(\nabla \times\ \mathbf{A}) \times \mathbf{n}\Big|_{\partial\Omega} &= \mathbf{0}, \\ \mathbf{A} \cdot \mathbf{n}\Big|_{\partial\Omega} = \frac{\partial\phi}{\partial n}\Big|_{\partial\Omega} &= 0, \\ \int_\Omega \phi dV = 0. \end{aligned}$$

[Haber & Ascher '00]

Elements of a finite volume discretization

- $\mathbf{A}, \ \boldsymbol{\nabla}\phi, \ \mathbf{s}$ at face centers (like $\mathbf{E}$)
- $\phi, \nabla \cdot \mathbf{A}$ at cell center (like m)
- $\widehat{\sigma}$ by harmonic averaging at face centers
- μ by arithmetic averaging at edge centers (like $\mathbf{H}$)



One cell in a staggered grid

Solving the linear algebra equations

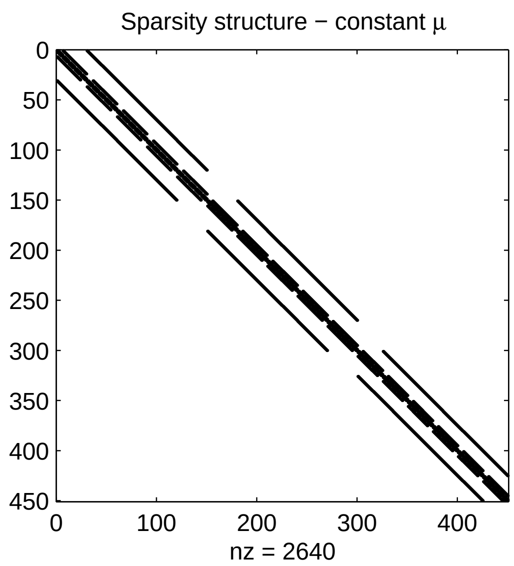
For $\mu = \mu_0$ obtain

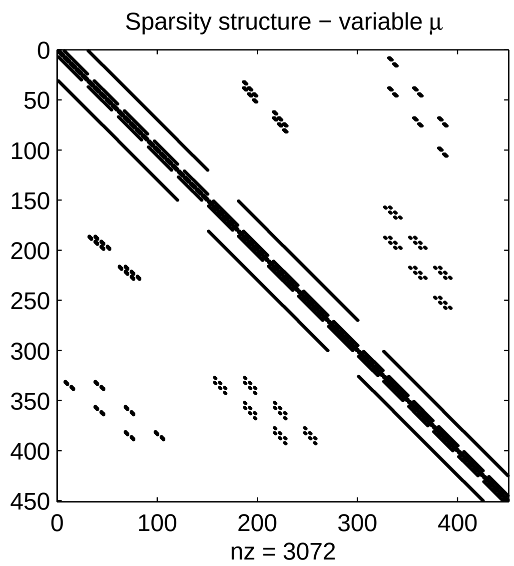
$$\mathcal{L}u = b$$

$$\mathcal{L} := \begin{pmatrix} H_1 & & & L_1 \\ & H_2 & & L_2 \\ & & H_3 & L_3 \\ D_1 & D_2 & D_3 & H_4 \end{pmatrix}, \quad u := \begin{pmatrix} A^{(1)} \\ A^{(2)} \\ A^{(3)} \\ \phi \end{pmatrix}, \ b := \begin{pmatrix} b^{(1)} \\ b^{(2)} \\ b^{(3)} \\ b^{(4)} \end{pmatrix}.$$

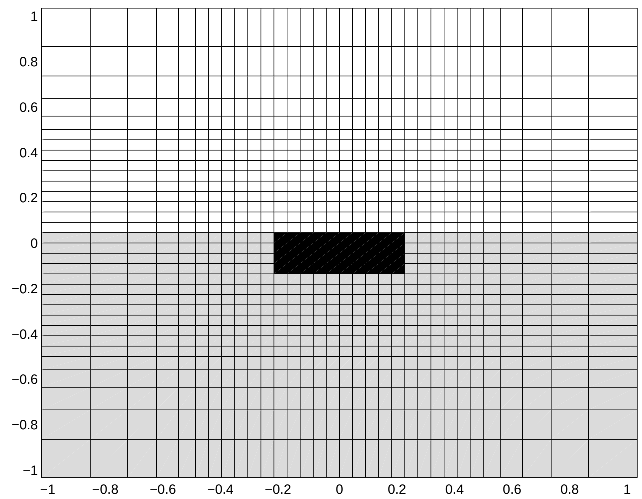
H_i are the dominant blocks.

Apply BICGSTAB with a block preconditioner: block ILU(t) (with tolerane 10^{-2}) or one cycle of multigrid.


Sparsity structure − constant μ
0
50
100
150
200
250
300
350
400
450
0
100
200
300
400
nz = 2640


Sparsity structure − variable μ
0
50
100
150
200
250
300
350
400
450
0
100
200
300
400
nz = 3072

Example: magnetic and electric sources



A cube of conductivity σ_c and permeability μ_c is embedded inside a homogeneous earth with conductivity σ_e and permeability μ_e. Also, in the air $\widehat{\sigma} = -\imath\omega\epsilon_0$ and $\mu = \mu_0$.

$\mu_0 = 4\pi \cdot 10^{-7}$ H/m, $\epsilon_0 = 8.85 \cdot 10^{-12}$ F/m, $\sigma_e = 10^{-3}$ S/m, $\mu_e = \mu_0$.

Try a magnetic source (div-free) and an electric dipole source (not div-free).

ω	Electric source				Magnetic source			
	iterations		operations		iterations		operations	
	$\mathbf{A}, \phi$	$\mathbf{E}$	$\mathbf{A}, \phi$	$\mathbf{E}$	$\mathbf{A}, \phi$	$\mathbf{E}$	$\mathbf{A}, \phi$	$\mathbf{E}$
10^0	46	9856	3.8	640	77	86	5.0	5.6
10^2	66	10323	4.3	670	82	91	5.3	5.9
10^4	77	10878	5.0	706	89	103	5.8	6.7
10^6	97	11212	6.3	728	99	113	6.4	7.3

Grid: 32^3 cells; $\tilde{\sigma} = 10^3$, $\tilde{\mu} = 10^2$. Our method $(\mathbf{A}, \phi)$ vs traditional Yee's method (applied to $\mathbf{E}$).

$\tilde{\sigma} = \sigma_c/\sigma_e \quad \tilde{\mu} = \mu_c/\mu_e$

	$\tilde{\mu} = \tilde{\sigma} = 10$		$\tilde{\mu} = \tilde{\sigma} = 1000$	
Grid size	ILU	SSOR	ILU	SSOR
8^3	6	20	58	268
16^3	10	32	108	453
32^3	23	52	213	789
64^3	31	76	396	1302

Iteration counts for different grids, for two sets of problem coefficients and using two preconditioners. Here $\omega = 10^2$.

Multigrid preconditioning

Use one multigrid cycle for each of the blocks H_i.

- Although each block is centered differently, can use a black box: Dendy's BOXMG. (Thanks, Joel Dendy and David Moulton !!). Use vertex-based coarsening, one $V(2,1)$ cycle. Red-black Gauss-Seidel or alternating plane relaxation.

- Fourier analysis: obtain convergence speed independent of grid size.

- In practice expect the multigrid preconditioner to become more effective than ILU when grid is finer than, say, 40^3.

Uniform grids							
σ_c (S/m)	# of cells	$\omega = 10^0$Hz		$\omega = 10^2$Hz		$\omega = 10^4$Hz	
		$\mathrm{M_M}$	$\mathrm{M_I}$	$\mathrm{M_M}$	$\mathrm{M_I}$	$\mathrm{M_M}$	$\mathrm{M_I}$
10^{-2}	10^3	3	7	3	7	6	11
	20^3	2	13	2	13	5	22
	30^3	2	20	3	20	5	34
	40^3	3	27	3	27	5	46
	50^3	3	33	3	33	5	61
10^0	10^3	3	9	3	9	6	12
	20^3	3	18	3	19	5	35
	30^3	3	30	3	30	5	56
	40^3	3	41	3	41	5	66
	50^3	3	44	3	44	5	100
10^2	10^3	3	9	3	9	6	12
	20^3	3	20	3	20	6	37
	30^3	3	30	3	27	6	55
	40^3	3	40	3	40	6	76
	50^3	3	49	3	48	6	97

BiCGStab iterations with an electric dipole source and uniform grids.

The discrete point of view

We have reformulated Maxwell's equations and then discretize them.

Given a mimic-type discretization, one can view the process from a linear algebra point of view.

Generate the matrices $\boldsymbol{\nabla}_h, \;\; \nabla_h \times\, , \;\; \nabla_h^2$

Easy to verify that

$$
\begin{aligned}
\nabla_h \times \; \boldsymbol{\nabla}_h &= 0 \\
\boldsymbol{\nabla}_h^T \nabla_h \times\, ^T &= 0 \\
\nabla_h \times\, ^T \nabla_h \times + \boldsymbol{\nabla}_h \boldsymbol{\nabla}_h^T &= \nabla_h^2
\end{aligned}
$$

The discrete point of view

Maxwell's equations for $\mathbf{E}_h$ are

$$(\nabla_h \times {}^T\nabla_h \times -\imath\omega S)\mathbf{E}_h = q$$

Set $\mathbf{E}_h = \mathbf{A}_h + \boldsymbol{\nabla}_h \phi_h; \quad \boldsymbol{\nabla}_h^T \mathbf{A}_h = 0$

Use the above vector identities

$$(\nabla_h \times {}^T \nabla_h \times -\imath\omega S)\mathbf{E}_h \;=\; q$$

$$(\nabla_h \times {}^T \nabla_h \times -\imath\omega S)(\mathbf{A}_h + \boldsymbol{\nabla}_h \phi_h) \;=\; q$$

$$\boldsymbol{\nabla}_h^T \mathbf{A}_h = 0$$

$$(\nabla_h \times {}^T \nabla_h \times +\boldsymbol{\nabla}_h \boldsymbol{\nabla}_h^T - \imath\omega S)\mathbf{A}_h + \imath\omega S \boldsymbol{\nabla}_h \phi_h \;=\; q$$

$$\boldsymbol{\nabla}_h^T \mathbf{A}_h = 0$$

$$
\begin{aligned}
(\nabla_h^2 - \imath\omega S)\mathbf{A}_h + \imath\omega S \boldsymbol{\nabla}_h \phi_h &= q \\
\boldsymbol{\nabla}_h^T (S\mathbf{A}_h + S\boldsymbol{\nabla}_h \phi_h) = \boldsymbol{\nabla}_h^T q
\end{aligned}
$$

Application

- If you have a mimic code with $\mathbf{E}, \mathbf{H}$ and you want to run it faster without getting to MG you can discretly reformulate and use standard iterative techniques and preconditioners with reasonable success.

- Your code is still $\mathcal{O}(h^2)$ accurate

- If you want to use MG, you can use the simple MG on the Laplacians rather than on the whole problem.

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