Math 346, HW8 Solution

5.1.2

(a,b) We need to verify that for \( \lambda \geq 0 \) up to certain upper limit, the linear program:

Minimize \( 16x_1 + 14x_2 \)
subject to \( 10x_1 + 4x_2 \geq 124 + 2\lambda \)
\( 3x_1 + 5x_2 \geq 60 - \lambda \)
\( x_1, x_2 \geq 0. \)

has the same optimal solution with the LP when \( \lambda = 0 \).

From the textbook, we know that \((x_1, x_2) = (10, 6)\) is the optimal solution with optimum value 244, therefore the final simplex tableau looks like this (\(x_3, x_4\) are the slack variables, so \(B\) is formed by taking the two columns (10, 3) and (4, 5), for which we can compute \(B^{-1}\)):

\[
\begin{bmatrix}
1 & 0 & 5/38 & -2/19 & 10 \\
0 & 1 & -3/38 & 5/19 & 6 \\
0 & 0 & c_3 & c_4 & -244
\end{bmatrix}
\]

When we change \(b = (124, 60)\) to \((124 + 2\lambda, 60 - \lambda)\), the \(B^{-1}b\) is equal to

\[
\begin{bmatrix}
10 \\
6
\end{bmatrix} + \begin{bmatrix}
5/38 & -2/19 & 2\lambda \\
-3/38 & 5/19 & -\lambda
\end{bmatrix} = \begin{bmatrix}
10 + 7/19 \cdot \lambda \\
6 - 8/19 \cdot \lambda
\end{bmatrix}
\]

As long as it is nonnegative, the daily minimum cost is equal to 16\((10 + 7/19 \cdot \lambda) + 14(6 - 8/19 \cdot \lambda) = 244\) and remains unchanged, solving \(6 - 8/19 \cdot \lambda \geq 0\) gives \(\lambda \leq 57/4\).

5.1.3

We consider the linear program (suppose the daily requirement for nutritional element A increases by \(\lambda_1\), that of B increases by \(\lambda_2\))

Minimize \( 16x_1 + 14x_2 \)
subject to \( 10x_1 + 4x_2 \geq 124 + \lambda_1 \)
\( 3x_1 + 5x_2 \geq 60 + \lambda_2 \)
\( x_1, x_2 \geq 0. \)
Similarly as before, $B^{-1}b$ is equal to $(10 + 5/38 \cdot \lambda_1 - 2/19 \cdot \lambda_2, 6 - 3/38 \cdot \lambda_1 + 5/19 \cdot \lambda_2)$. The objective function is equal to

$$16(10 + 5/38 \cdot \lambda_1 - 2/19 \cdot \lambda_2) + 14(6 - 3/38 \cdot \lambda_1 + 5/19 \cdot \lambda_2) = 244 + \lambda_1 + 2\lambda_2.$$ 

If $\lambda_2 > 0$ (we increase the requirement for element $B$), then the cost increases by $2\lambda_1$. So for each 10-unit increase, the cost increases by 20 cents which is more than the increase of the value (15 cents), so this won’t do any good. But if $\lambda_1 > 0$, then the cost increases by only 10 cents while the value of the stock increases by 15 cents.

To determine the upper limit for increasing $\lambda_1$, we set $\lambda_2 = 0$ and let $B^{-1}b \geq 0$, we have $10 + 5/38 \cdot \lambda_1 \geq 0$, and $6 - 3/38 \cdot \lambda_1 \geq 0$, solving them gives $\lambda_1 \leq 76$. So this works for up to 76 units of increment for the element $A$. Another way is to solve the following LP (the objective function is cost minus increase in value)

$$\text{Minimize } 16x_1 + 14x_2 - 1.5\lambda_1$$

subject to

$$10x_1 + 4x_2 \geq 124 + \lambda_1$$
$$3x_1 + 5x_2 \geq 60$$
$$x_1, x_2, \lambda_1 \geq 0.$$

The solution is $(x_1, x_2, \lambda_1) = (20, 0, 76)$, which means that the producer can increase the requirement for element $A$ by at most 76, and during this procedure the value of stock minus the cost increases.

5.3.6

The final tableau for the dual LP is:

$$\begin{bmatrix} y_1 & y_2 & y_3 & y_4 & y_5 \\ y_2 & 0 & 1 & -3/2 & 1/4 & -3/8 & 1 \\ y_1 & 1 & 0 & 9/2 & -1/4 & 7/8 & 1 \\ 0 & 0 & 72 & 6 & 21 & 144 \end{bmatrix}$$

From the tableau, we know that the optimal solution is $(y_1, y_2, y_3) = (1, 1, 0)$, with $y_1, y_2$ being the basic variable. Now we would like to fix the nutritional requirement for $B$ and $C$, and check for which range of nutritional requirement for $A$, the optimal solution of the dual remains the same. In other words, we change the vector $b$ in the dual:

$$\text{Maximize } b^T y$$

subject to

$$A^T y \leq c$$
$$y \geq 0,$$

and we hope to keep the optimal solution. Note that after we change 60 to $\lambda$, the last row of the tableau becomes

$$-\begin{bmatrix} \lambda, 84, 72, 0, 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & -3/2 & 1/4 & -3/8 \end{bmatrix} = \begin{bmatrix} 0, 0, 9\lambda/2 - 198, 21 - \lambda/4, 7\lambda/8 - 63/2 \end{bmatrix}$$
This vector is nonnegative when $44 \leq \lambda \leq 84$.

Now suppose we fix requirements for $A$ and $C$ and change requirement for $B$ from 84 to $\lambda$, then we have the last row being:

$$-\begin{bmatrix}60, \lambda, 72, 0, 0\end{bmatrix} + [\lambda, 60] \begin{bmatrix}0 & 1 & -3/2 & 1/4 & -3/8 \\ 1 & 0 & 9/2 & -1/4 & 7/8\end{bmatrix} \geq 0$$

We can solve $60 \leq \lambda \leq 132$.

Again, if we fix $A$ and $B$ and change the requirement for $C$ from 72 to $\lambda$, we have the last row being:

$$-\begin{bmatrix}60, 84, \lambda, 0, 0\end{bmatrix} + [84, 60] \begin{bmatrix}0 & 1 & -3/2 & 1/4 & -3/8 \\ 1 & 0 & 9/2 & -1/4 & 7/8\end{bmatrix} \geq 0$$

Solving this we get $\lambda \leq 144$.

5.5.3

In the linear program:

Maximize $11x_1 + 4x_2 + x_3 + 15x_4$
subject to $3x_1 + x_2 + 2x_3 + 4x_4 \leq 28$
$8x_1 + 2x_2 - x_3 + 7x_4 \leq 50$
$x_1, x_2, x_3, x_4 \geq 0$.

By simplex method, the final tableau looks like (from Page 184):

$$\begin{bmatrix}
 x_4 & x_2 & x_3 & x_4 & x_5 & x_6 \\
-2 & 0 & 5 & 1 & 2 & -1 \\
11 & 1 & -18 & 0 & -7 & 4 \\
3 & 0 & 2 & 0 & 2 & 1 \\
& & & & & 6 \\
& & & & & 4 \\
& & & & & 106
\end{bmatrix}$$

Note that by taking the fourth column and second column in the original LP, the matrix

$$B = \begin{bmatrix}4 & 1 \\ 7 & 2\end{bmatrix}.$$ 

We can compute its inverse, which gives:

$$B^{-1} = \begin{bmatrix}2 & -1 \\ -7 & 4\end{bmatrix}.$$ 

Suppose we change 28 to $28 + \lambda$, note that this only changes $\vec{b}$, the new $B^{-1}\vec{b}$ is equal to

$$\begin{bmatrix}2 & -1 \\ -7 & 4\end{bmatrix} \cdot \begin{bmatrix}28 + \lambda \\ 50\end{bmatrix} = \begin{bmatrix}6 + 2\lambda \\ 4 - 7\lambda\end{bmatrix}.$$ 

If $B^{-1}\vec{b} \geq \vec{0}$, then this remains the optimal solution, and the basis remains to be $\{x_4, x_2\}$. Solving the inequality gives $-3 \leq \lambda \leq 4/7$. This optimal solution gives a maximum that is equal to

$$11x_1 + 4x_3 + x_3 + 15x_4 = 4(4 - 7\lambda) + 15(6 + 2\lambda) = 2\lambda + 106.$$