Strong Duality of Linear Programming

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For an $m \times n$ matrix $A$, a vector $c \in \mathbb{R}^n$ and another vector $b \in \mathbb{R}^m$. Given a linear program in its maximization form:

Maximize $c^T x$

subject to $Ax \leq b$ \hspace{1cm} (Primal)

$x \geq 0$

Its dual linear program is:

Minimize $b^T y$

subject to $A^T y \geq c$ \hspace{1cm} (Dual)

$y \geq 0$

The weak duality says that if $x_0$ is a feasible solution to the primal, and $y_0$ is a feasible solution to the dual, then $c^T x_0 \leq b^T y_0$. In other words, suppose both linear programs have an optimal solution, then the maximum of the primal linear program is always less or equal to the minimum of the dual linear program.

For linear programming, we actually have the following strong duality (which might not be true for other types of problems like integer programming or non-linear programming): if the dual (resp. primal) has an optimal solution $y_0$ (resp. $x_0$), then the primal (resp. dual) has an optimal solution $x_0$ (resp. $y_0$), and moreover $c^T x_0 = b^T y_0$. There are several different proofs of the strong duality. For example one proof is based on the simplex method (see Theorem 4.4.2, and Exercise 4.4.8). Here in the next few pages I will show another proof using the Farkas Lemma.

**Farkas Lemma:** Let $A$ be an $m \times n$ matrix, and $b$ be an $m$-dimensional vector, then exactly one of the following two statements is true:

(i) There exists $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, such that $Ax = b$ and $x \geq 0$.

(ii) There exists $y = (y_1, \cdots, y_m) \in \mathbb{R}^m$, such that $A^T y \geq 0$ and $b^T y < 0$.

The Farkas Lemma can be understood from a geometric perspective. Suppose $A = [a_1, \cdots, a_n]$, where $a_1, \cdots, a_n$ are the column vectors of $A$. Then (i) in the lemma says that

$$b = Ax = x_1 a_1 + x_2 a_2 + \cdots + x_n a_n$$
for some nonnegative $x_1, \cdots, x_n$. Geometrically it means that the vector $b$ is contained in the convex cone $C$ formed by the nonnegative linear combinations of the column vectors $a_1, a_2, \cdots, a_n$. Part (ii) in the Lemma translates to that there exists a $m$-dimensional vector $y$, such that $a_i^T y \geq 0$ for $i = 1, 2, \cdots, n$ and $b^T y < 0$. If we take $y$ to be the vector perpendicular to a hyperplane passing through the origin in $R^m$, then $a_i^T y \geq 0$ for all $i$ and $b^T y < 0$ essentially says that the vector $b$ and the convex cone $C$ are on different sides of this hyperplane. So the Farkas Lemma says that either the vector is contained in the cone (part (i)), or there exists a hyperplane separating the vector from the cone (part (ii)). This should give a satisfying geometrical explanation for the Farkas Lemma by now, and we defer a mathematically rigorous proof to the end of this note.

The Farkas Lemma in the above form is elegant but for the purpose of proving the strong duality, the following form is a bit easier to use:

**Another form of Farkas Lemma**: Let $A$ be an $m \times n$ matrix, and $b$ be an $m$-dimensional vector, then exactly one of the following two statements is true:

1. There exists $x = (x_1, \cdots, x_n) \in R^n$, such that $Ax \geq b$ and $x \geq 0$.
2. There exists $y = (y_1, \cdots, y_m) \in R^m$, such that $A^T y \geq 0$, $b^T y < 0$, and $y \leq 0$.

To see that this form is equivalent to the first form of Farkas Lemma. Now we prove this form from the Farkas Lemma, the other direction is also similar. Suppose (1) is false, then by converting this linear program into the standard form, we can see that

$$Ax - Ix_s = b$$
$$x, x_s \geq 0$$

has no feasible solution. Here $x_s$ is the vector formed by the slack variables added to the original LP in (1) for obtaining the standard form, and $I$ is the $m \times m$ identity matrix. We can apply the first form of Farkas Lemma, with $A$ replaced by the matrix $[A \mid I]$. It follows that (from (ii)) there exists $y$ such that

$$\begin{bmatrix} A^T \\ -I \end{bmatrix} y \geq 0, \quad b^T y < 0$$

This implies that $A^T y \geq 0$, $y \leq 0$, and $b^T y < 0$. Therefore (2) must be true. Similarly, one can show that if (1) is true, then (2) is false.

Now we are ready to prove the strong duality theorem using the second form of Farkas Lemma.

**Proof of the Strong Duality**: Let $y_0$ be the optimal solution to the dual. We will prove by contradiction: suppose the conclusion of the strong duality is false, we will construct a new feasible solution to the dual which beats $y_0$. There are two possible scenarios for the conclusion to fail: (1) the primal linear program is infeasible; (2) the primal linear program is feasible, but the optimal solution
$x_0$ has $b^Ty_0 \neq c^T x_0$ (from the weak duality, we have $b^Ty_0 \geq c^T x_0$, so this implies $b^Ty_0 > c^T x_0$). For both cases, the following linear program is infeasible:

\[
\begin{align*}
Ax &\leq b \\
c^T x &\geq b^Ty_0 \\
x &\geq 0
\end{align*}
\]

Writing in a matrix form, there does not exist $x \geq 0$, such that

\[
\begin{bmatrix} -A \\ c^T \end{bmatrix} x \geq \begin{bmatrix} -b \\ b^Ty_0 \end{bmatrix}
\]

By the second form of Farkas Lemma, there exists $y \in \mathbb{R}^m$ and $w \in \mathbb{R}$, such that

\[
\begin{bmatrix} -A^T \\ c \end{bmatrix} \begin{bmatrix} y \\ w \end{bmatrix} \geq 0, \quad \begin{bmatrix} -b \\ b^Ty_0 \end{bmatrix} \begin{bmatrix} y \\ w \end{bmatrix} < 0, \quad \begin{bmatrix} y \\ w \end{bmatrix} \leq 0.
\]

Let $y = w \cdot z$, where $z$ is an $m$-dimensional vector, using the fact that $w \leq 0$, we have (after dividing everything by $w$)

\[
A^T z \geq c, \quad b^T z < b^Ty_0, \quad z \geq 0.
\]

Therefore we have found a feasible solution $z$ to the dual which gives a smaller objective value than $y_0$. This contradicts the optimality of $y_0$.

**Proof of the Farkas Lemma:** Suppose (i) is false, so there does not exist $x$ such that $Ax = b$ and $x \geq 0$, which means that the vector $b$ is outside the cone $C$ formed by the nonnegative linear combinations of columns of $A$. We consider the projection of $b$ on the cone $C$, denoted by $p$. In other words, $p = Aw$, where $w \geq 0$, is the point in the cone $C$ that is closest to $b$. One key observation is that for any $x \in C$,

\[
\langle b - p, x - p \rangle \leq 0. \tag{1}
\]

To see this, let $d = x - p$, we need to show that $\langle d, p - b \rangle \geq 0$. This is because for every $\lambda \geq 0$, by the convexity of $C$, $p + \lambda d$ is still in $C$, so the distance between $p + \lambda d$ and $b$ must be at least the distance between $p$ and $b$. Hence we have for every $\lambda \geq 0$,

\[
\langle p + \lambda d - b, p + \lambda d - b \rangle \geq \langle p - b, p - b \rangle.
\]

Simplifying the inequality, we get

\[
2\lambda \langle d, p - b \rangle + \lambda^2 \langle d, d \rangle \geq 0.
\]

Since this inequality holds for all $\lambda \geq 0$, it is not hard to see that $\langle d, p - b \rangle \geq 0$ (otherwise we can choose $\lambda$ to be a sufficiently small positive number to get a contradiction).
From (1), since every $x \in C$ can be written as $Ay$ with $y \geq 0$, therefore for every $y \geq 0$,

$$\langle b - Aw, Ay - Aw \rangle \leq 0.$$  \hspace{1cm} (2)

Let $z = Aw - b$, then for every $y \geq 0$,

$$\langle A^T z, y - w \rangle = \langle z, Ay - Aw \rangle \geq 0.$$

Take $y = w + e_i \geq 0$, where $e_i$ is the unit vector with 1 in the $i$-th coordinate and 0 in all the other coordinates. Then from $\langle A^T z, e_i \rangle \geq 0$, we have $A^T z \geq 0$. Moreover,

$$\langle b, z \rangle = \langle b, Aw - b \rangle = \langle Aw, Aw - b \rangle - \langle Aw - b, Aw - b \rangle.$$

Note that in (2), taking $y = 0$ gives $\langle b - Aw, Aw \rangle \geq 0$. Therefore $\langle Aw, Aw - b \rangle \leq 0$, and clearly we have $\langle Aw - b, Aw - b \rangle > 0$ since $b$ is outside the cone, so the distance between $b$ and $p = Aw$ is strictly positive. Combining these two inequalities, it follows that $\langle b, z \rangle < 0$. Therefore we have found a vector $z \in \mathbb{R}^m$ such that $A^T z \geq 0$ and $b^T z = \langle b, z \rangle < 0$, so (ii) is true.

Now assume that (i) is true, then there exists $x \geq 0$ such that $Ax = b$, therefore for every $y$ with $A^T y \geq 0$, we have

$$b^T y = (Ax)^T y = x^T A^T y \geq 0.$$

So there does not exist a vector $y$ such that both $A^T y \geq 0$ and $b^T y < 0$ are true. \hfill \Box