5.5.3

In the linear program:

Maximize \( 11x_1 + 4x_2 + x_3 + 15x_4 \)
subject to
\[
\begin{align*}
3x_1 + x_2 + 2x_3 + 4x_4 & \leq 28 \\
8x_1 + 2x_2 - x_3 + 7x_4 & \leq 50 \\
x_1, x_2, x_3, x_4 & \geq 0.
\end{align*}
\]

By simplex method, the final tableau looks like (from Page 184):

\[
\begin{bmatrix}
-2 & 0 & 5 & 1 & 2 & -1 & 6 \\
11 & 1 & -18 & 0 & -7 & 4 & 4 \\
3 & 0 & 2 & 0 & 2 & 1 & 106
\end{bmatrix}
\]

Note that by taking the fourth column and second column in the original LP, the matrix
\[
B = \begin{bmatrix}
4 & 1 \\
7 & 2
\end{bmatrix}.
\]

We can compute its inverse, which gives:
\[
B^{-1} = \begin{bmatrix}
2 & -1 \\
-7 & 4
\end{bmatrix}.
\]

Suppose we change 28 to 28 + \( \lambda \), note that this only changes \( \vec{b} \), the new \( B^{-1}\vec{b} \) is equal to
\[
\begin{bmatrix}
2 & -1 \\
-7 & 4
\end{bmatrix} \begin{bmatrix}
28 + \lambda \\
50
\end{bmatrix} = \begin{bmatrix}
6 + 2\lambda \\
4 - 7\lambda
\end{bmatrix}.
\]

If \( B^{-1}\vec{b} \geq 0 \), then this remains the optimal solution, and the basis remains to be \( \{x_4, x_2\} \). Solving the inequality gives \(-3 \leq \lambda \leq 4/7 \). This optimal solution gives a maximum that is equal to
\[
11x_1 + 4x_3 + x_3 + 15x_4 = 4(4 - 7\lambda) + 15(6 + 2\lambda) = 2\lambda + 106.
\]

5.6.1
In the step 2 of the dual simplex algorithm, suppose there exists $r$ such that $b_r < 0$, and $a_{rj} \geq 0$ for all $j$, then if we consider the $r$-th constraint, it looks like:

$$a_{r1}x_1 + a_{r2}x_2 + \cdots + a_{rn}x_n = b_r.$$ 

However, if all $a_{rj} \geq 0$, then the left hand side of this equality is nonnegative for a feasible solution (since all $x_i$’s need to be nonnegative in a feasible solution), while the right hand side is equal to $b_r$ which is strictly negative, contradiction. Therefore this system of linear constraints has no feasible solution.

6.2.5

Figure 6.3:

subject to $5x_1 + 4x_2 - 20 \leq M_1(1 - y_1)$  
$3x_1 + 8x_2 - 24 \leq M_2(1 - y_2)$  
$y_1 + y_2 \geq 1$  
$x_1, x_2 \geq 0; y_1, y_2 \in \{0, 1\}$

Now let’s determine the value of $M_1, M_2$. Note that for any solution $(x_1, x_2)$ in the feasible region, one always have $0 \leq x_1 \leq 8$, and $0 \leq x_2 \leq 5$. Therefore

$$5x_1 + 4x_2 - 20 \leq 5 \times 8 + 4 \times 5 - 20 = 40.$$ 

One can choose $M_1$ to be any real number greater or equal to 40.

For $M_2$, note that

$$3x_1 + 8x_2 - 24 \leq 3 \times 8 + 8 \times 5 - 24 = 40.$$ 

So $M_2$ can be chosen as any real number at least 40.

Figure 6.4:

subject to $x_1 + x_2 - 1 \leq M_1(1 - y_1)$  
$3x_1 + x_2 - 3 \geq M_2(1 - y_2)$  
$3x_1 + 4x_2 - 12 \leq M_3(1 - y_2)$  
$y_1 + y_2 \geq 1$  
$x_1, x_2 \geq 0; y_1, y_2 \in \{0, 1\}$

Note that any feasible solution $(x_1, x_2)$ satisfy $0 \leq x_1 \leq 4$, and $0 \leq x_2 \leq 3$. Therefore

$$x_1 + x_2 - 1 \leq 4 + 3 - 1 = 6$$ 
$$3x_1 + x_2 - 3 \geq 3 \times 0 + 0 - 3 = -3$$ 
$$3x_1 + 4x_2 - 12 \leq 3 \times 4 + 4 \times 3 - 12 = 12$$

Therefore one can choose $M_1$ to be any real at least 6, $M_2$ to be any real at most $-3$, $M_3$ to be any real at least 12. Note that the constraint $y_1 + y_2 \geq 1$ can also be changed to $y_1 + y_2 = 1$, since the two shaded triangles only intersect trivially.
Figure 6.5:

subject to \(2x_1 + 5x_2 - 10 \leq M_1(1 - y_1)\)
\(x_1 - 3 \leq M_2(1 - y_2)\)
\(x_1 - x_2 \geq M_3(1 - y_2)\)
\(y_1 + y_2 \geq 1\)
\(x_1, x_2 \geq 0; y_1, y_2 \in \{0, 1\}\)

Note that any feasible solution \((x_1, x_2)\) satisfy \(0 \leq x_1 \leq 5\), and \(0 \leq x_2 \leq 3\).
Therefore
\(2x_1 + 5x_2 - 10 \leq 2 \times 5 + 5 \times 3 - 10 = 15\)
\(x_1 - 3 \leq 5 - 3 = 2\)
\(x_1 - x_2 \geq 0 - 3 = -3\).

So we can choose \(M_1\) to be any real at least 15, \(M_2\) be any real at least 2, and \(M_3\) be any real at most \(-3\).

6.2.16

We let \(y_i = 1\) if the \(i\)-th constraint is satisfied, and let \(y_i = 0\) if the \(i\)-th constraint is not satisfied. Now we can write down the integer programming (we will pick the constant \(M_i\)'s later):

Maximize \(9x_1 + 8x_2 + 7x_3\)
subject to \(x_1 + x_2 + x_3 \leq 500\)
\(3x_1 - 3x_2 + 4x_3 - 1000 \leq M_1(1 - y_1)\)
\(x_1 - 2x_3 - 200 \geq -M_2(1 - y_2)\)
\(x_1 + x_2 - 300 \leq M_3(1 - y_3)\)
\(x_1 + x_2 - 300 \geq -M_4(1 - y_3)\)
\(y_1 + y_2 + y_3 \geq 2\)
\(y_1, y_2, y_3 \leq 1\)
\(x_1, x_2, x_3, y_1, y_2, y_3 \geq 0; y_1, y_2, y_3\) integral.

Next we will decide the value for \(M_i\)'s. For example, we know that for all \(x_i\), they lie in the interval \([0, 500]\) from the first inequality. Therefore,

\(3x_1 - 3x_2 + 4x_3 - 1000 \leq 3 \times 500 - 3 \times 0 + 4 \times 500 - 1000 = 2500\),

which means that if we pick \(M_1 = 2501\), then the first inequality is automatically satisfied in the case \(y_1 = 0\) (we want no extra constraint in this case). Similarly since \(x_1 - 2x_3 - 200 \geq 0 - 2 \times 500 - 200 = -1700\), we can pick \(M_2 = 1800\), and determine the value for \(M_3\) and \(M_4\) as well.

6.4.1(a)
We would like to solve the following integer program using the Branch-and-Bound algorithm:

\[(IP1) \text{ Maximize } z = 5x_1 + 2x_2 \]
\[\text{ subject to } \begin{align*}
6x_1 + 2x_2 &\leq 13 \\
-6x_1 + 7x_2 &\leq 14 \\
x_1, x_2 &\geq 0 \text{ and integral.}
\end{align*}\]

Using simplex method, we know that the corresponding LP has an optimal solution \((7/6, 3)\) that gives \(z = 71/6\). Now we consider two new integer programs by taking \(x_1 \leq 1\) and \(x_1 \geq 2\), respectively. For the first IP:

\[\text{Maximize } z = 5x_1 + 2x_2 \]
\[\text{ subject to } \begin{align*}
6x_1 + 2x_2 &\leq 13
\end{align*}\]
\[ (IP2a) \]
\[\begin{align*}
-6x_1 + 7x_2 &\leq 14 \\
x_1 &\leq 1 \\
x_1, x_2 &\geq 0 \text{ and integral.}
\end{align*}\]

The corresponding LP has an optimal solution \((1, 20/7)\) that gives \(z = 75/7\). The second IP:

\[\text{Maximize } z = 5x_1 + 2x_2 \]
\[\text{ subject to } \begin{align*}
6x_1 + 2x_2 &\leq 13
\end{align*}\]
\[ (IP2b) \]
\[\begin{align*}
-6x_1 + 7x_2 &\leq 14 \\
x_1 &\geq 2 \\
x_1, x_2 &\geq 0 \text{ and integral.}
\end{align*}\]

corresponds to a LP which has an optimal solution \((2, 1/2)\) that gives \(z = 11\).

Now we continue to branch from these two IPs, IP2a has two branches using \(x_2 \leq 2\) and \(x_2 \geq 3\) respectively:

\[\text{Maximize } z = 5x_1 + 2x_2 \]
\[\text{ subject to } \begin{align*}
6x_1 + 2x_2 &\leq 13 \\
-6x_1 + 7x_2 &\leq 14 \\
x_1 &\leq 1 \\
x_2 &\leq 2 \\
x_1, x_2 &\geq 0 \text{ and integral.}
\end{align*}\]
\[ (IP3a) \]

Its corresponding LP has an optimal solution \((1, 2)\) that gives \(z = 9\), for this branch the algorithm stops here since we already arrive at an integral optimal solution.

The second branch of IP2a is:

\[\text{Maximize } z = 5x_1 + 2x_2 \]
\[\text{ subject to } \begin{align*}
6x_1 + 2x_2 &\leq 13 \\
-6x_1 + 7x_2 &\leq 14 \\
x_1 &\leq 1 \\
x_2 &\geq 3 \\
x_1, x_2 &\geq 0 \text{ and integral.}
\end{align*}\]
\[ (IP3b) \]
Its corresponding LP is infeasible.

Now for IP2b, we also have two branches according to $x_2 \leq 0$ (meaning that $x_2 = 0$), or $x_2 \geq 1$:

\[
\begin{align*}
\text{Maximize} & \quad z = 5x_1 + 2x_2 \\
\text{subject to} & \quad 6x_1 + 2x_2 \leq 13 \\
& \quad -6x_1 + 7x_2 \leq 14 \\
& \quad x_1 \geq 2 \\
& \quad x_2 = 0 \\
& \quad x_1, x_2 \geq 0 \text{ and integral.}
\end{align*}
\]

The corresponding LP has an optimal solution $(13/6, 0)$ that gives $65/6$. The second branch:

\[
\begin{align*}
\text{Maximize} & \quad z = 5x_1 + 2x_2 \\
\text{subject to} & \quad 6x_1 + 2x_2 \leq 13 \\
& \quad -6x_1 + 7x_2 \leq 14 \\
& \quad x_1 \geq 2 \\
& \quad x_2 \geq 1 \\
& \quad x_1, x_2 \geq 0 \text{ and integral.}
\end{align*}
\]

corresponds to a LP that is infeasible.

Now we start the branching process from IP3c, by setting $x_1 \leq 2$ (since already $x_1 \geq 2$, we have $x_1 = 2$) and $x_1 \geq 3$:

\[
\begin{align*}
\text{Maximize} & \quad z = 5x_1 + 2x_2 \\
\text{subject to} & \quad 6x_1 + 2x_2 \leq 13 \\
& \quad -6x_1 + 7x_2 \leq 14 \\
& \quad x_1 = 2 \\
& \quad x_2 = 0 \\
& \quad x_1, x_2 \geq 0 \text{ and integral.}
\end{align*}
\]

The LP has a unique optimal solution $(2, 0)$ that gives $z = 10$ (which also updates the current best value 9). For

\[
\begin{align*}
\text{Maximize} & \quad z = 5x_1 + 2x_2 \\
\text{subject to} & \quad 6x_1 + 2x_2 \leq 13 \\
& \quad -6x_1 + 7x_2 \leq 14 \\
& \quad x_1 \geq 3 \\
& \quad x_2 = 0 \\
& \quad x_1, x_2 \geq 0 \text{ and integral.}
\end{align*}
\]

Its corresponding LP is infeasible. Therefore we finish the branch-and-bound algorithm and conclude that the optimal solution is $(x_1, x_2) = (2, 0)$, which gives optimum value $z = 10$ (same with what graphical method gives in problem 6.1.1(b)).