On almost k-covers of hypercubes

Alexander Clifton * Hao Huang †

June 24, 2019

Abstract

In this paper, we consider the following problem: what is the minimum number of affine hyperplanes in $\mathbb{R}^n$, such that all the vertices of $\{0,1\}^n \backslash \{\vec{0}\}$ are covered at least k times, and $\vec{0}$ is uncovered? The $k = 1$ case is the well-known Alon-Füredi theorem which says a minimum of n affine hyperplanes is required, proved by the Combinatorial Nullstellensatz.

We develop an analogue of the Lubell-Yamamoto-Meshalkin inequality for subset sums, and completely solve the fractional version of this problem, which also provides an asymptotic answer to the integral version for fixed n and $k \to \infty$. We also use a Punctured Combinatorial Nullstellensatz developed by Ball and Serra, to show that a minimum of $n + 3$ affine hyperplanes is needed for $k = 3$, and pose a conjecture for arbitrary k and large n.

1 Introduction

Alon's Combinatorial Nullstellensatz [1] is one of the most powerful algebraic tools in modern combinatorics. Alon and Füredi [2] used this method to prove the following elegant result: any set of affine hyperplanes that covers all the vertices of the n-cube $Q^n = \{0,1\}^n$ but one contains at least n affine hyperplanes. There are many generalizations and analogues of this theorem: for rectangular boxes [2], Desarguesian affine and projective planes [6, 7], quadratic surfaces and Hermitian varieties in $PG(n, q)$ [4]. The common theme of these results are: in many point-line (point-surface) geometries, to cover all the points except one, more lines are needed than to cover all points.

*Department of Mathematics, Emory University, Atlanta, USA. Email: alexander.james.clifton@emory.edu. Research supported in part by a George W. Woodruff Fellowship.

†Department of Mathematics, Emory University, Atlanta, USA. Email: hao.huang@emory.edu. Research supported in part by the Collaboration Grants from the Simons Foundation.

In this paper, we consider the following generalization of the Alon-Füredi theorem. Let $f(n,k)$ be the minimum number of affine hyperplanes needed to cover every vertex of Q^n at least k times (except for $\vec{0} = (0, \cdots, 0)$ which is not covered at all). For convenience, from now on we call such a cover an *almost k-cover* of the n-cube. The Alon-Füredi theorem gives $f(n,1) = n$ since the affine hyperplanes $x_i = 1$, for $i = 1, \cdots, n$ covers $Q^n \backslash \{\vec{0}\}$. Their result also leads to $f(n,2) = n+1$. The lower bound follows from observing that when removing one hyperplane from an almost 2-cover, the remaining hyperplanes form an almost 1-cover. On the other hand, the n affine hyperplanes $x_i = 1$, together with $x_1 + \cdots + x_n = 1$ cover every vertex of $Q^n \backslash \{\vec{0}\}$ at least twice.

These observations immediately lead to a lower bound $f(n,k) \geqslant n+k-1$ by removing $k-1$ affine hyperplanes, and an upper bound $f(n,k) \leqslant n + \binom{k}{2}$ by considering the following almost k-cover: $x_i = 1$ for $i = 1, \cdots, n$, together with $k-t$ copies of $\sum_{i=1}^{n} x_i = t$, for $t = 1, \cdots, k-1$. In this construction, every binary vector with t 1-coordinates is covered t times by $\{x_i = 1\}$, and $k-t$ times by $x_1 + \cdots + x_n = t$. The total number of hyperplanes is $n + \sum_{t=1}^{k-1}(k-t) = n + \binom{k}{2}$.

Note that for $k = 3$, the inequalities above give $n+2 \leqslant f(n,3) \leqslant n+3$. We used a punctured version of the Combinatorial Nullstellensatz, developed by Ball and Serra [3] to show that the upper bound is tight in this case. We also improve the lower bound for $k \geqslant 4$.

Theorem 1.1. *For $n \geqslant 2$,*

$$f(n,3) = n+3.$$

For $k \geqslant 4$ and $n \geqslant 3$,

$$n+k+1 \leqslant f(n,k) \leqslant n + \binom{k}{2}.$$

Our second result shows that for fixed n and the multiplicity $k \to \infty$, the aforementioned upper bound $f(n,k) \leqslant n + \binom{k}{2}$ is indeed far from being tight. Indeed $f(n,k) \sim c_n k$ when $k \to \infty$. Note that $f(n,k)$ is the optimum of an integer program. We consider the following linear relaxation of it: we would like to assign to every affine hyperplane H in $\mathbb{R}^n$ a non-negative weight $w(H)$, with the constraints

$$\sum_{\vec{v} \in H} w(H) \geqslant k, \quad \text{for every } \vec{v} \in Q^n \backslash \{\vec{0}\},$$

and

$$\sum_{\vec{0} \in H} w(H) = 0,$$

such that $\sum_H w(H)$ is minimized. Such an assignment w of weights is called a *fractional almost k-cover* of Q^n. Denote by $f^*(n,k)$ the minimum of $\sum_H w(H)$, i.e. the minimum size of a fractional almost k-cover. We are able to determine the precise value of $f^*(n,k)$ for every value of n and k.

Theorem 1.2. *For every n and k,*

$$f^*(n,k) = \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) k.$$

It implies that for fixed n and $k \to \infty$,

$$f(n,k) = \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} + o(1)\right) k,$$

which grows linearly in k.

As an intermediate step of proving Theorem 1.2, we proved the following theorem, which can be viewed as an analogue of the well-known Lubell-Yamamoto-Meshalkin inequality [5, 8, 9, 10] for subset sums. Moreover the inequality is tight for all non-zero binary vectors $\vec{a} = (a_1, \cdots, a_n)$.

Theorem 1.3. *Given n real numbers $a_1, \cdots, a_n$, let*

$$\mathcal{A} = \{S : \varnothing \neq S \subset [n], \sum_{i \in S} a_i = 1\}.$$

Then

$$\sum_{S \in \mathcal{A}} \frac{1}{|S|\binom{n}{|S|}} \leqslant 1.$$

Equivalently, let $\mathcal{A}_t = \{S : S \in \mathcal{A}, |S| = t\}$, then

$$\sum_{t=1}^{n} \frac{|\mathcal{A}_t|}{t\binom{n}{t}} \leqslant 1.$$

The rest of the paper is organized as follows. In the next section, we resolve the almost 3-cover case, and show that the answer to the almost 4-cover problem has only two possible values, thus proving Theorem 1.1. Section 3 contains the proofs of Theorems 1.2 and 1.3. The final section contains some concluding remarks and open problems.

2 Almost 3-covers of the n-cube

The following Punctured Combinatorial Nullstellensatz was proven by Ball and Serra (Theorem 4.1 in [3]). Let $\mathbb{F}$ be a field and f be a non-zero polynomial in $\mathbb{F}[x_1, \cdots, x_n]$. We say $\vec{a} = (a_1, \cdots, a_n)$ is a *zero of multiplicity* t of f, if t is the minimum degree of the terms that occur in $f(x_1 + a_1, \cdots, x_n + a_n)$.

Lemma 2.1. *For $i = 1, \cdots, n$, let $D_i \subset S_i \subset \mathbb{F}$ and $g_i = \prod_{s \in S_i}(x_i - s)$ and $\ell_i = \prod_{d \in D_i}(x_i - d)$. If f has a zero of multiplicity at least t at all the common zeros of $g_1, \cdots, g_n$, except at least one point of $D_1 \times \cdots \times D_n$ where it has a zero of multiplicity* *less than t, then there are polynomials h_τ satisfying* $\deg(h_\tau) \leqslant \deg(f) - \sum_{i\in\tau}\deg(g_i)$, *and a non-zero polynomial u satisfying* $\deg(u) \leqslant \deg(f) - \sum_{i=1}^{n}(\deg(g_i) - \deg(\ell_i))$, *such that*

$$f = \sum_{\tau\in T(n,t)} g_{\tau(1)}\cdots g_{\tau(t)} h_\tau + u\prod_{i=1}^{n}\frac{g_i}{\ell_i}.$$

Here $T(n,t)$ indicates the set of all non-decreasing sequences of length t on $[n]$.

This punctured Nullstellensatz will be our main tool in proving Theorem 1.1. We start with the $k = 3$ case.

Theorem 2.2. *For $n \geqslant 2$, $f(n,3) = n + 3$.*

Proof. To show that $f(n,3) = n+3$, it suffices to establish the lower bound. We prove by contradiction. Suppose $H_1, \cdots, H_{n+2}$ are $n+2$ affine hyperplanes that form an almost 3-cover of Q^n. Without loss of generality, assume the equation defining H_i is $\langle \vec{b}_i, \vec{x}\rangle = 1$, for some non-zero vector $\vec{b}_i \in \mathbb{R}^n$. Define $P_i = \langle \vec{b}_i, \vec{x}\rangle - 1$, and let

$$f = P_1 P_2 \cdots P_{n+2}.$$

Since $H_1, \cdots, H_{n+2}$ form an almost 3-cover of Q^n, every binary vector $\vec{x} \in Q^n\backslash\{\vec{0}\}$ is a zero of multiplicity at least 3 of the polynomial f. We apply Lemma 2.1 with

$$D_i = \{0\},\quad S_i = \{0,1\},\quad g_i = x_i(x_i-1),\quad \ell_i = x_i,$$

and write f in the following form:

$$f = \sum_{1\leqslant i\leqslant j\leqslant k\leqslant n} x_i(x_i-1)x_j(x_j-1)x_k(x_k-1)h_{ijk} + u\prod_{i=1}^{n}(x_i-1),$$

with $\deg(u) \leqslant \deg(f) - n = 2$.

Note that $f = 0$ on $Q^n\backslash\{\vec{0}\}$. Moreover,

$$\frac{\partial f}{\partial x_i} = \sum_{j=1}^{n+2} P_1\cdots P_{j-1}\cdot\frac{\partial P_j}{\partial x_i}\cdot P_{j+1}\cdots P_{n+2}.$$

Recall that P_j is a polynomial of degree 1, thus $\partial f/\partial x_i$ is just a linear combination of $P_1\cdots\hat{P}_j\cdots P_{n+2}$. Note that removing a single hyperplane still gives an almost 2-cover. Therefore $\partial f/\partial x_i$ vanishes on $Q^n\backslash\{\vec{0}\}$. One can similarly show that all the second order partial derivatives of f vanish on $Q^n\backslash\{\vec{0}\}$ as well. More generally, if f is the product of equations of the affine hyperplanes from an almost k-cover, then all the j-th order derivatives of f vanish on $Q^n\backslash\{\vec{0}\}$, for $j = 0, \cdots, k-1$. It is not hard to observe that $x_i(x_i-1)x_j(x_j-1)x_k(x_k-1)h_{ijk} = g_i g_j g_k h_{ijk}$ also has its t-th order partial derivatives vanishing on the entire cube Q^n, for $t \in \{0,1,2\}$, since $x_i(x_i-1)=0$ on Q^n. Therefore the following polynomial

$$h = u\prod_{i=1}^{n}(x_i-1)$$

has j-th order partial derivatives vanishing on $Q^n\backslash\{\vec{0}\}$, for $j=0,1,2$.

We denote by e_i the n-dimensional unit vector with the i-th coordinate being 1. By calculations,

$$\frac{\partial h}{\partial x_i} = \frac{\partial u}{\partial x_i}\prod_{j=1}^{n}(x_j-1) + u\prod_{j\neq i}(x_j-1).$$

Therefore

$$0 = \frac{\partial h}{\partial x_i}(e_i) = (-1)^{n-1}u(e_i),$$

and this implies

$$u(e_i) = 0 \quad \text{for } i=1,\cdots,n.$$

Furthermore,

$$\frac{\partial^2 h}{\partial x_i^2} = \frac{\partial^2 u}{\partial x_i^2}\prod_{j=1}^{n}(x_j-1) + 2\frac{\partial u}{\partial x_i}\prod_{j\neq i}(x_j-1).$$

Therefore

$$0 = \frac{\partial^2 h}{\partial x_i^2}(e_i) = (-1)^{n-1}\cdot 2\frac{\partial u}{\partial x_i}(e_i),$$

and this implies

$$\frac{\partial u}{\partial x_i}(e_i) = 0 \quad \text{for } i=1,\cdots,n.$$

Finally,

$$\frac{\partial^2 h}{\partial x_i x_j} = \frac{\partial^2 u}{\partial x_i x_j}\prod_{k=1}^{n}(x_k-1) + \frac{\partial u}{\partial x_i}\prod_{k\neq j}(x_k-1) + \frac{\partial u}{\partial x_j}\prod_{k\neq i}(x_k-1) + u\prod_{k\neq i,j}(x_k-1)$$

By evaluating it on e_i and e_i+e_j, we have

$$\frac{\partial u}{\partial x_j}(e_i) = u(e_i) = 0, \quad \text{and} \quad u(e_i+e_j)=0.$$

Summarizing the above results u is a polynomial of degree at most 2, satisfying: (i) $u=0$ at e_i and e_i+e_j; (ii) $\partial u/\partial x_i = 0$ at e_j (possible to have $i=j$). We define a new single-variable polynomial w,

$$w(x) = u(x\cdot e_i + e_j).$$

Then $\deg(w) \leqslant 2$, and $w(0) = w(1) = w'(0) = 0$, which implies $w \equiv 0$. Let

$$u = \sum_i a_{ii}x_i^2 + \sum_{i<j} a_{ij}x_ix_j + \sum_i b_ix_i + c.$$

This gives for all $i \neq j$,

$$a_{ii} = 0, \quad a_{ij} + b_i = 0, \quad a_{ii} + b_i + c = 0.$$

On other other hand $\partial u/\partial x_i = 0$ at e_i gives

$$2a_{ii} + b_i = 0.$$

It is not hard to derive from these equalities that

$$a_{ii} = a_{ij} = b_i = c = 0.$$

Therefore $u \equiv 0$. But then we have $f(\vec{0}) = 0$, which contradicts the assumption that $\vec{0}$ is not covered by any of the $n+2$ affine hyperplanes. Therefore $f(n,3) = n+3$ for $n \geqslant 2$. Note that $f(1,3) = 3$ and the proof does not work for $n = 1$ because $e_i + e_j$ does not exist in a 1-dimensional space. □

Note that Theorem 2.2 already implies $f(n,4) \geqslant f(n,3)+1 = n+4$ for all $n \geqslant 2$. For $n = 2$, it is straightforward to check that $f(2,4) = 6$, with an optimal almost 4-cover $x_1 = 1$ (twice), $x_2 = 1$ (twice), and $x_1 + x_2 = 1$ (twice). However for $n \geqslant 3$, we can improve this lower bound by 1.

Theorem 2.3. *For $n \geqslant 3$, $f(n,4) \in \{n+5, n+6\}$. Moreover, for $3 \leqslant n \leqslant 5$, $f(n,4) = n+5$.*

Proof. Suppose $n \geqslant 3$, we would like to prove by contradiction that $n+4$ affine hyperplanes cannot form an almost 4-cover of Q^n. Following the notations in the previous proof, we have

$$P_1 \cdots P_{n+4} = f = \sum_{1 \leqslant i \leqslant j \leqslant k \leqslant l \leqslant n} g_ig_jg_kg_lh_{ijkl} + u\prod_{i=1}^n (x_i - 1),$$

with $\deg(u) \leqslant 4$. Following similar calculations, u satisfies the following relations: (i) $u = 0$ at e_i, $e_i + e_j$ and $e_i + e_j + e_k$ for distinct i, j, k; (ii) $\partial u/\partial x_i = 0$ at e_j and $e_j + e_k$ for distinct j, k ($i = j$ or $i = k$ possible); (iii) $\partial^2 u/\partial x_i^2 = 0$ at e_j ($i = j$ possible); (iv) $\partial^2 u/\partial x_i \partial x_j = 0$ at e_k ($i = k$ or $j = k$ possible). Suppose

$$u = \sum a_{iiii}x_i^4 + \sum a_{iiij}x_i^3x_j + \cdots + \sum b_{iii}x_i^3 + \cdots + \sum c_{ii}x_i^2 + \cdots + \sum d_ix_i + e.$$

Since $f(\vec{0}) = (-1)^{n+4} = (-1)^n$, we know that $u(\vec{0}) = 1$ and thus $e = 1$.

Let $w(x) = u(x \cdot e_i + e_j)$. Then $w(0) = w(1) = w'(0) = w'(1) = w''(0) = 0$. Since $w(x)$ has degree at most 4, we immediately have $w \equiv 0$. This gives

$$a_{iiii} = 0, \tag{1}$$

$$a_{iiij} + b_{iii} = 0. \tag{2}$$

$$a_{iijj} + b_{iij} + c_{ii} = 0. \tag{3}$$

$$a_{ijjj} + b_{ijj} + c_{ij} + d_i = 0. \tag{4}$$

$$a_{jjjj} + b_{jjj} + c_{jj} + d_j + 1 = 0 \tag{5}$$

Using $u(e_i) = 0$, $\partial u/\partial x_i(e_i) = 0$ and $\partial^2 u/\partial x_i^2(e_i) = 0$, we have

$$a_{iiii} + b_{iii} + c_{ii} + d_i + 1 = 0,$$

$$4a_{iiii} + 3b_{iii} + 2c_{ii} + d_i = 0,$$

$$12a_{iiii} + 6b_{iii} + 2c_{ii} = 0.$$

Using $a_{iiii} = 0$, we can solve this system of linear equations and get $b_{iii} = -1$, $c_{ii} = 3$, $d_i = -3$. This implies $a_{iiij} = 1$. Plugged into the equations (3) and (4), we have:

$$a_{iijj} + b_{iij} = -3,$$

$$b_{iij} + c_{ij} = 2.$$

Now using $\partial^2 u/\partial x_i \partial x_j(e_i) = 0$, we have $3a_{iiij} + 2b_{iij} + c_{ij} = 0$, which gives

$$2b_{iij} + c_{ij} = -3.$$

The three linear equations above give $b_{iij} = -5$, $c_{ij} = 7$, $a_{iijj} = 2$.

For $n \geqslant 3$, we can also utilize the relation $\partial^2 u/(\partial x_i \partial x_j) = 0$ at e_k. This gives $a_{ijkk} + b_{ijk} + c_{ij} = 0$, hence

$$a_{ijkk} + b_{ijk} = -7.$$

Also $\partial u/(\partial x_i) = 0$ at $e_j + e_k$ simplifies to

$$a_{ijkk} + a_{ijjk} + b_{ijk} + 3 = 0.$$

Together they give $b_{ijk} = -11$ and $a_{ijkk} = 4$. Finally, by calculations

$$\begin{aligned} u(e_i + e_j + e_k) &= 3a_{iiii} + 6a_{iiij} + 3a_{iijj} + 3a_{iijk} + 3b_{iii} + 6b_{iij} + b_{ijk} + 3c_{ii} + 3c_{ij} + 3d_i + e \\ &= 2 \neq 0. \end{aligned}$$

This gives a contradiction. Therefore for $n \geqslant 3$, there is no u of degree at most 4 satisfying the aforementioned relations. This shows for $n \geqslant 3$, $f(n, 4) \geqslant n + 5$. The proof does not work for $n < 3$ because $e_i + e_j + e_k$ does not exist in a 1-dimensional or 2-dimensional space. Since $f(n,4) \leqslant n + \binom{4}{2} = n+6$, it can only be either $n+5$ or $n+6$, proving the first claim in Theorem 2.3.

To show that $f(n,4) = n+5$ for $3 \leqslant n \leqslant 5$, we only need to construct almost 4-covers of Q^n using $n+5$ affine hyperplanes. For Q^3, note that $x_1 = 1$, $x_2 = 1$, $x_3 = 1$, and $x_1 + x_2 + x_3 = 1$ form an almost 2-cover. Doubling it gives an almost 4-cover of Q^3 with 8 affine hyperplanes. For Q^4, the following 9 affine hyperplanes form an almost 4-cover: $x_1 = 1$, $x_2 = 1$, $x_3 = 1$, $x_4 = 1$, $x_1 + x_4 = 1$, $x_2 + x_4 = 1$, $x_3 + x_4 = 1$, $x_1 + x_2 + x_3 = 1$, $x_1 + x_2 + x_3 + x_4 = 1$. For Q^5, one can take $x_i = 1$ for $i = 1, \cdots 5$, together with $x_i + x_{i+1} + x_{i+2} = 1$ for $i = 1, \cdots, 5$, where the addition is in $\mathbb{Z}_5$. □

Now we can combine these two results we just obtained to prove Theorem 1.1.

Proof of Theorem 1.1. The $k = 3$ case has been resolved by Theorem 2.2. On the other hand we have

$$f(n,k) \geqslant f(n,k-1) + 1,$$

since removing an affine hyperplane from an almost k-cover gives an almost $(k-1)$-cover. Therefore for $k \geqslant 4$ and $n \geqslant 3$,

$$f(n,k) \geqslant f(n,4) + (k-4) \geqslant n + 5 + (k-4) = n + k + 1.$$

The upper bound follows from the construction in the introduction. □

3 Fractional almost k-covers of the n-cube

In this section, we determine $f^*(n,k)$ precisely and prove Theorem 1.2. We first establish an upper bound by an explicit construction of almost k-covers.

Lemma 3.1. *(i) For every n, k,*

$$f^*(n,k) \leqslant \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) k.$$

(ii) When k is divisible by nx, with $x = lcm(\binom{n-1}{0}, \binom{n-1}{1}, \cdots, \binom{n-1}{n-1})$, we have

$$f(n,k) \leqslant \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) k.$$

Proof. For (ii), it suffices to show that when $k = nx$, we can find an almost k-cover of Q^n, using $k(1 + 1/2 + \cdots + 1/n)$ hyperplanes. We can then replicate this process to upper bound $f(n,k)$ where k is any multiple of nx.

For $j = 1, \cdots, n$, we will use every affine hyperplane of the form $x_{i_1} + x_{i_2} + \cdots + x_{i_j} = 1$ a total of $\frac{nx}{j\binom{n}{j}}$ times. This number is actually an integer since it is equal to $\frac{x}{\binom{n-1}{j-1}}$, and by definition, x is divisible by all $\binom{n-1}{j-1}$.

There are $\binom{n}{j}$ affine hyperplanes in this form, so the total number of being used is

$$\sum_{j=1}^{n} \frac{nx}{j\binom{n}{j}} \cdot \binom{n}{j} = \sum_{j=1}^{n} \frac{nx}{j} = \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) k$$

This is the number of hyperplanes claimed. If we could show that they form an almost nx-cover of Q^n, then we can scale the weights by a constant factor to obtain a fractional almost k-cover of Q^n for every k and (i) follows immediately.

Now we must show that these affine hyperplanes cover each point the appropriate number of times. It is apparent that $(0, \cdots, 0)$ is never covered. Because of the symmetric nature of our construction, we just need to check how many times we have covered a vertex that has t ones as coordinates. It gets covered by $t\binom{n-t}{j-1}$ distinct hyperplanes of the form $x_{i_1} + x_{i_2} + \cdots + x_{i_j} = 1$, each of which appears $\frac{nx}{j\binom{n}{j}}$ times. Thus, the total number of times a point with t ones is covered is given by:

$$\begin{aligned}
\sum_{j=1}^{n} \frac{nx}{j\binom{n}{j}} \cdot t\binom{n-t}{j-1} &= nxt \sum_{j=1}^{n} \frac{\binom{n-t}{j-1}}{j\binom{n}{j}} = nxt \sum_{j=1}^{n} \frac{(n-t)!(n-j)!}{(n-t-j+1)!n!} \\
&= nxt \cdot \frac{(n-t)!}{n!} \cdot \sum_{j=1}^{n} \frac{(n-j)!}{(n-t-j+1)!} \\
&= \frac{nx}{(t-1)!\binom{n}{t}} \sum_{j=1}^{n} \frac{(n-j)!}{(n-t-j+1)!} = \frac{nx}{\binom{n}{t}} \sum_{j=1}^{n} \binom{n-j}{t-1} \\
&= \frac{nx}{\binom{n}{t}} \binom{n}{t} = nx = k
\end{aligned}$$

□

To establish the lower bound in Theorem 1.2, first we assign weights to each vertex of Q^n we wish to cover. A vertex with t ones as coordinates is given weight $\frac{1}{t\binom{n}{t}}$. Then the sum of the weights of all the vertices is:

$$\sum_{t=1}^{n} \binom{n}{t} \cdot \frac{1}{t\binom{n}{t}} = \sum_{t=1}^{n} \frac{1}{t}$$

So if we cover each vertex k times, the sum over all affine hyperplanes of the weights of the vertices they cover is $k(1 + 1/2 + \cdots + 1/n)$. Thus, if we can show that no hyperplane can cover a set of vertices whose weights sum to more than 1, we will have proven the lower bound. Given an affine hyperplane H not containing $\vec{0}$, denote by $\mathcal{A}_t$ the set of vertices with t ones covered by H. We wish to prove Theorem 1.3, i.e.

$$\sum_{t=1}^{n} \frac{|\mathcal{A}_t|}{t\binom{n}{t}} \leqslant 1.$$

In general, vertices of $Q^n \backslash \{\vec{0}\}$ correspond to nonempty subsets of $[n]$. It is worth noting that if the equation of H is $a_1x_1 + \cdots + a_nx_n = 1$, and all coefficients a_i are strictly positive, the subsets corresponding to the vertices it covers will form an antichain. By the Lubell-Yamamoto-Meshalkin inequality,

$$\sum_{t=1}^{n} \frac{|\mathcal{A}_t|}{t\binom{n}{t}} \leqslant \sum_{t=1}^{n} \frac{|\mathcal{A}_t|}{\binom{n}{t}} \leqslant 1.$$

However, some coefficients a_i may be non-positive. In order to consider a more general hyperplane, we will associate each vertex it covers to some permutations of $[n]$. Consider the vertex $(c_1, c_2, \cdots, c_n) \in Q^n$ where the coordinates which are ones are $c_{i_1}, \cdots, c_{i_t}$. We will associate this vertex to the permutations, $(d_1, d_2, \cdots, d_n)$ of $[n]$ which begin with $\{i_1, i_2, \cdots, i_t\}$ in some order and also have $\sum_{k=1}^{j} a_{d_k} < 1$ for $1 \leqslant j < t$.

Lemma 3.2. *No permutation of $[n]$ is associated to more than one vertex on the same hyperplane.*

Proof. Suppose for the sake of contradiction that a permutation is associated to two vertices, v and w, of the same hyperplanes. They may have either the same or a different number of ones as coordinates.

Suppose that v and w both have a ones as coordinates. The permutations associated to v have the a indices where v has a 1 as their first a entries and the permutations associated to v will have the a indices where w has a 1 as their first a entries. However, v and w do not have their ones in the exact same places so the set of the first a entries is not the same for any pair of a permutation associated to v and a permutation associated to w.

We are left to consider the case where v and w do not have the same number of ones as coordinates. Without loss of generality, v has a ones as coordinates and w has b ones as coordinates where $a > b$. Suppose the permutation associated to both of them begins with $(d_1, d_2, \cdots, d_b)$. By the restrictions on permutations associated to v, we have that $\sum_{j=1}^{b} a_{d_j} < 1$. However, the conditions on permutations associated to w tell us that $(d_1, d_2, \cdots, d_b)$ are precisely the indices where w has a 1 coordinate. This implies $\sum_{j=1}^{b} a_{d_j} = 1$, giving a contradiction.

□

Lemma 3.3. *The total number of permutations associated to a vertex with t ones as coordinates is at least $(t-1)!(n-t)!$*

Proof. There are $(n-t)!$ ways to arrange the indices other than $\{i_1, i_2, \cdots, i_t\}$, so it suffices to show that there exist at least $(t-1)!$ ways to order $\{i_1, i_2, \cdots, i_t\}$ as $(d_1, d_2, \cdots, d_t)$ such that we have $\sum_{k=1}^{j} a_{d_k} < 1$ for $1 \leqslant j < t$. We notice that $(t-1)!$ is the number of ways to order $\{i_1, i_2, \cdots, i_t\}$ around a circle (up to rotations, but not reflections). Thus it suffices to show that for each circular ordering of $\{i_1, i_2, \cdots, i_t\}$, we can choose a starting place from which we may continue clockwise and label the elements as $(d_1, d_2, \cdots, d_t)$ in such a way that $\sum_{k=1}^{j} a_{d_k} < 1$ for all $1 \leqslant j < t$.

Equivalently, the values of a_{i_k}, which happen to sum to 1, have been listed around a circle for $1 \leqslant k \leqslant t$. We wish to find some starting point from which all the partial sums of up to $t-1$ terms from that point are less than 1. We can subtract $1/t$ from each to give the equivalent problem of t numbers, which sum to 0, written around a circle and needing to find a starting point from which all the partial sums of $1 \leqslant j \leqslant t-1$ terms are less than $1 - \frac{j}{t}$. It suffices to find a starting point for which the aforementioned partial sums are at most 0.

Consider all possible sums of any number of consecutive terms along the circle and choose the largest. We will label the terms in this sum as $e_1, e_2, \cdots, e_m$ and continue to order clockwise around the circle $e_{m+1}, e_{m+2}, \cdots, e_t$. Choose the starting point to be e_{m+1}. If any of the partial sums $e_{m+1} + e_{m+2} + \cdots + e_{m+j}$ exceeds 0, for $m + j \leqslant t$, we could have simply chosen $e_1, e_2, \cdots, e_{m+j}$ to get a larger sum than $e_1 + e_2 + \cdots + e_m$. Similarly, if $e_{m+1} + e_{m+2} + \cdots + e_t + e_1 + e_2 + \cdots + e_j > 0$ for some $1 \leqslant j < m$, then we can note that $(e_1 + e_2 + \cdots + e_t) + (e_1 + e_2 + \cdots + e_j)$ exceeds $e_1 + e_2 + \cdots + e_m$, and since $e_1 + e_2 + \cdots + e_t = 0$, we have that $e_1 + e_2 + \cdots + e_j > e_1 + e_2 + \cdots + e_m$, a contradiction. Thus, if we start at e_{m+1} and move clockwise around the circle, the first $t-1$ partial sums will be at most 0, as desired.

□

Combining the previous results, we prove Theorem 1.3, which can be viewed as an analogue of the LYM inequality for partial sums.

Proof of Theorem 1.3. By definition, sets in $\mathcal{A}$ correspond to vertices of Q^n covered by the hyperplane H with equation $a_1x_1 + \cdots + a_nx_n = 1$. From Lemma 3.2 and 3.3, these vertices define disjoint collections of permutations of length n. Moreover if $S \in \mathcal{A}$ has size t then there are at least $(t-1)!(n-t)!$ permutations associated to it. Since in total there are at most $n!$ permutations, we get

$$\sum_{S \in \mathcal{A}} (|S| - 1)!(n - |S|)! \leqslant n!,$$

which implies

$$\sum_{S \in \mathcal{A}} \frac{1}{|S|\binom{n}{|S|}} \leqslant 1$$

as desired. □

Now we are ready to prove our main theorem in this section.

Proof of Theorem 1.2. As mentioned before, we assign weight $\frac{1}{t\binom{n}{t}}$ to a vertex of $Q^n \backslash \{\vec{0}\}$ with t ones as coordinates. By Lemma 1.3, every affine hyperplane covers a set of vertices whose weights sum to at most 1. Therefore in an optimal fractional almost k-cover $\{w(H)\}$,

$$f^*(n,k) = \sum_H w(H) \geqslant k \cdot \sum_{t=1}^{n} \frac{\binom{n}{t}}{t\binom{n}{t}} = \left(\sum_{i=1}^{n} \frac{1}{i}\right) k.$$

With the upper bound proved in Lemma 3.1, we have

$$f^*(n,k) = \left(\sum_{i=1}^{n} \frac{1}{i}\right) k.$$

For integral almost k-covers, note that $f(n,k) \geqslant f^*(n,k)$. Using Lemma 3.1 again,

$$f(n,k) = f^*(n,k) = \left(\sum_{i=1}^{n} \frac{1}{i}\right) k,$$

whenever nx divides k. For fixed n and $k \to \infty$, note that $f(n,k)$ is monotone in k, which immediately implies

$$f(n,k) = \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} + o(1)\right) k.$$

□

For small values of n, we can actually determine the value of $f(n,k)$ for every k. It seems that for large k, $f(n,k)$ is not far from its lower bound $\lceil f^*(n,k) \rceil$. Trivially $f(1,k) = k$.

Theorem 3.4. *The following statements are true:*
(i) $f(2,k) = \lceil \frac{3k}{2} \rceil$ *for* $k \geqslant 1$.
(ii) $f(3,k) = \lceil \frac{11k}{6} \rceil$ *for* $k \geqslant 2$ *and* $f(3,1) = 3$.

Proof. (i) From previous discussions, there exists an almost 2-cover of Q^2 using 3 affine hyperplanes. Therefore $f(2,k+2) \leqslant f(2,k) + 3$, and it suffices to check $f(2,1) = 2$ and $f(2,2) = 3$ which are both obvious.
(ii) There exists an almost 6-cover of Q^3 using 11 affine hyperplanes. Therefore $f(3,k+6) \leqslant f(3,k) + 11$. It suffices to check $f(3,k) \leqslant \lceil \frac{11k}{6} \rceil$ for $k = 2, \cdots, 5$ and $k = 7$. From $f(n,2) = n+1$, we have $f(3,2) = 4$. $f(3,3) \leqslant 6$ follows from Theorem 1.1. $f(3,4) \leqslant 8$ since $f(3,4) \leqslant 2f(3,2)$. $f(3,5) \leqslant 10$ by taking each of $x_i = 1$ twice, $x_1 + x_2 + x_3 = 1$ three times, and $x_1 + x_2 + x_3 = 2$ once. $f(3,7) \leqslant 13$ follows from taking each of $x_1 = 1$, $x_2 = 1$, $x_3 = 1$, $x_1 + x_2 = 1$, $x_1 + x_3 = 1$ twice, and $x_2 + x_3 = 1$, $x_2 + x_3 - x_1 = 1$, $x_1 + x_2 + x_3 = 1$ once. □

With the assistance of a computer program, we also checked that $f(4,k) = \lceil \frac{25k}{12} \rceil$ for $k \geqslant 2$. $f(5,k) = \lceil \frac{137}{60}k \rceil$ for $k \geqslant 15$ except when $k \equiv 7 \pmod{60}$ where $f(5,k) = \lceil \frac{137}{60}k \rceil + 1$. The following question is natural.

Question 3.5. Does there exist an absolute constant $C > 0$ which does not depend on n, such that for a fixed integer n, there exists M_n, so that whenever $k \geqslant M_n$,

$$f(n,k) \leqslant \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) k + C?$$

If so, it would show that $f(n,k)$ and $f^*(n,k)$ differ by at most a constant when k is large.

4 Concluding Remarks

In this paper, we determine the minimum size of a fractional almost k-cover of Q^n, and find the minimum size of an integral almost k-cover of Q^n, for $k \leqslant 3$. Note that $f(n,1) = n$ for $n \geqslant 1$, $f(n,2) = n+1$ for $n \geqslant 1$, and $f(n,3) = n+3$ for $n \geqslant 2$. All of them attain the upper bound $f(n,k) \leqslant n + \binom{k}{2}$ whenever n is not too small. For larger k the following conjecture seems plausible.

Conjecture 4.1. *For an arbitrary fixed integer $k \geqslant 1$ and sufficiently large n,*

$$f(n,k) = n + \binom{k}{2}.$$

In other words, for large n, an almost k-cover of Q^n contains at least $n + \binom{k}{2}$ affine hyperplanes.

In particular, for $k = 4$, although $f(n,k) \leqslant n+5$ for $n \leqslant 5$, we suspect that for $n \geqslant 6$, $n+6$ affine hyperplanes are necessary for an almost 4-cover of Q^n. If we restrict our attention to almost k-covers of Q^n which use each of the affine hyperplanes $x_i = 1$ for $i = 1, \cdots, n$, we see that Conjecture 4.1, if true, will imply the following weaker conjecture:

Conjecture 4.2. *For fixed $k \geqslant 1$ and sufficiently large n, suppose $H_1, \cdots, H_m$ are affine hyperplanes in $\mathbb{R}^n$ not containing $\vec{0}$, and they cover all the vectors with t ones as coordinates at least $k-t$ times, for $t = 1, \cdots, k-1$. Then $m \geqslant \binom{k}{2}$.*

If this conjecture is true, then the $\binom{k}{2}$ bound is the best possible, since one can take $k-t$ copies of $x_1 + \cdots + x_n = t$ for $t = 1, \cdots, k-1$. We note that using our weights from earlier, and the fact that a hyperplane cannot cover vertices whose weights sum to more than 1, we require:

$$m \geqslant \sum_{t=1}^{k-1}(k-t)\binom{n}{t}\frac{1}{t\binom{n}{t}} = 1-k+\sum_{t=1}^{k-1}\frac{k}{t} = (1-o(1))k\ln k.$$

Remark added. Alon communicated to us that Conjecture 4.2 is true. With his permission, we include his proof using Ramsey-type arguments below. Let n be huge, and let S be a collection of m affine hyperplanes $H_1, \cdots, H_m$ satisfying the assumptions in Conjecture 4.2 and $N = [n]$. Color each subset of size $k-1$ by the index of the first hyperplane that covers it (m colors), by Ramsey there is a large subset N_1 of N so that all $(k-1)$-subsets of it are covered by the same hyperplane. Without loss of generality, the equation of this hyperplane is $\sum_i w_i x_i = 1$ and it follows that for all $j \in N_1$, all w_j are equal and hence all are equal $1/(k-1)$. Therefore this hyperplane cannot cover any $k-t$ subset of N_1 for $t \geqslant 2$. Now throw away this hyperplane and repeat the argument for subsets of size $k-2$ of N_1. Coloring each such subset by the pair of smallest two indices of the hyperplanes that cover it ($\binom{m}{2}$ colors), we get a monochromatic subset N_2 of N_1 and observe that here too each of these two hyperplanes whose equation is $\sum_i w_i x_i = 1$ has $w_j = 1/(k-2)$ for all $j \in N_2$. So these cannot be useful for covering smaller subsets of N_2, throw them away and repeat this process. After dealing with all subsets including those of size 1 we get the assertion of the conjecture. $\square$

Alon and Füredi [2] proved the following result using induction on $n-m$: for $n \geqslant m \geqslant 1$, then m hyperplanes that do not cover all vertices of Q^n miss at least 2^{n-m} vertices. Let $g(n,m,k)$ be the minimum number of vertices covered less than k times by m affine hyperplanes not passing through $\vec{0}$. The Alon-Füredi theorem shows $g(n,m,1) = 2^{n-m}$ for $m = 1, \cdots, n$. For $k = 2$, it is straightforward to show that for $m = 1, \cdots, n+1$, we have:

$$g(n,m,2) = 2^{n-m+1} \tag{6}$$

This is because $m-1$ hyperplanes leave at least 2^{n-m+1} vertices uncovered, and with one more hyperplane, these vertices cannot be covered twice. Similarly, for $k \geqslant 3$, we can obtain a trivial lower bound $g(n,m,k) \geqslant 2^{n-m+k-1}$. On the other hand, suppose $f(d,k) = t$ for $d \leqslant n$, then take the affine hyperplanes $H_1, \cdots, H_t$ in an almost k-cover of Q^d. Observe that $H_i \times \mathbb{R}^{n-d}$ is an affine hyperplane in Q^n not containing $\vec{0}$. It is easy to see that $\{H_i \times \mathbb{R}^{n-d}\}$ covers all the vertices of Q^n but those of the form $\vec{0} \times \{0,1\}^{n-d}$ at least k times. Therefore $g(n,t,k) \leqslant 2^{n-d}$. Theorem 1.1 shows $f(d,3) = d+3$ for $d \geqslant 2$, therefore $g(n,d+3,3) \leqslant 2^{n-d}$ or $g(n,m,3) \leqslant 2^{n-m+3}$ for $m \geqslant 5$. We believe that this upper bound is tight. Note that the trivial lower bound is $g(n,m,3) \geqslant 2^{n-m+2}$.

Conjecture 4.3.

$$g(n,m,3) = \begin{cases} 2^n, & m = 1, 2; \\ 2^{n-1}, & m = 3; \\ 2^{n-m+3}, & m = 4, \cdots, n+3. \end{cases}$$

One can further ask the following question for arbitrary k.

Question 4.4. Is it true that for all n, m, k,

$$g(n,m,k) = 2^{n-d},$$

where d is the maximum integer such that $f(d,k) \leqslant m$?

References

[1] N. Alon, Combinatorial Nullstellensatz, *Combinatorics, Probability and Computing*, **8** (1999), 7–29.

[2] N. Alon, Z. Füredi, Covering the cube by affine hyperplanes, *Eur. J. Comb.*, **14** (1993), 79–83.

[3] S. Ball, O. Serra, Punctured combinatorial Nullstellensätze, *Combinatorica*, **29** (5) (2009), 511–522.

[4] A. Blokhuis, A. Brouwer, T. Szőnyi, Covering all points except one, *Journal of Algebraic Combinatorics*, **32** (1) (2010), 59–66.

[5] B. Bollobás, On generalized graphs, *Acta Mathematica Academiae Scientiarum Hungaricae*, **16** (3-4) (1965) 447–452.

[6] A. Brouwer, A. Schrijver, The blocking number of an affine space, *J. Comb. Theory, Ser. A* **24** (1978), 251–253.

[7] R. Jamison, Covering finite fields with cosets of subspaces, *J. Comb. Theory, Ser. A*, **22** (1977), 253–266.

[8] D. Lubell, A short proof of Sperner's lemma, *J. Comb. Theory*, **1** (2) (1966), 299.

[9] L. Meshalkin, Generalization of Sperner's theorem on the number of subsets of a finite set, *Theory of Probability and Its Applications*, **8** (2) (1963), 203–204.

[10] K. Yamamoto, Logarithmic order of free distributive lattice, *Journal of the Mathematical Society of Japan*, **6** (1954), 343–353.