Nonnegative k-sums, fractional covers, and probability of small deviations

Noga Alon * Hao Huang † Benny Sudakov‡

Abstract

More than twenty years ago, Manickam, Miklós, and Singhi conjectured that for any integers n, k satisfying $n \geq 4k$, every set of n real numbers with nonnegative sum has at least $\binom{n-1}{k-1}$ k-element subsets whose sum is also nonnegative. In this paper we discuss the connection of this problem with matchings and fractional covers of hypergraphs, and with the question of estimating the probability that the sum of nonnegative independent random variables exceeds its expectation by a given amount. Using these connections together with some probabilistic techniques, we verify the conjecture for $n \geq 33k^2$. This substantially improves the best previously known exponential lower bound $n \geq e^{ck \log\log k}$. In addition we prove a tight stability result showing that for every k and all sufficiently large n, every set of n reals with a nonnegative sum that does not contain a member whose sum with any other $k-1$ members is nonnegative, contains at least $\binom{n-1}{k-1} + \binom{n-k-1}{k-1} - 1$ subsets of cardinality k with nonnegative sum.

1 Introduction

Let $\{x_1, \cdots, x_n\}$ be a set of n real numbers whose sum is nonnegative. It is natural to ask the following question: how many subsets of nonnegative sum must it always have? The answer is quite straightforward, one can set $x_1 = n-1$ and all the other $x_i = -1$, which gives 2^{n-1} subsets. This construction is also the smallest possible since for every subset A, either A or $[n]\backslash A$ or both must have a nonnegative sum. Another natural question is, what happens if we further restrict all the subsets to have a fixed size k? The same example yields $\binom{n-1}{k-1}$ nonnegative k-sums consisting of $n-1$ and $(k-1)$ -1's. This construction is similar to the extremal example in the Erdős-Ko-Rado theorem [8] which states that for $n \geq 2k$, a family of subsets of size k in $[n]$ with the property that every two subsets have a nonempty intersection has size at most $\binom{n-1}{k-1}$. However the relation between k-sum and k-intersecting family is somewhat subtle and there is no obvious way to translate one problem to the other.

Denote by $A(n, k)$ the minimum possible number of nonnegative k-sums over all possible choices of n numbers $x_1, \cdots, x_n$ with $\sum_{i=1}^n x_i \geq 0$. For which values of n and k, is the construction $x_1 =$

*Sackler School of Mathematics and Blavatnik School of Computer Science, Tel Aviv University, Tel Aviv 69978, Israel. Email: nogaa@tau.ac.il. Research supported in part by an ERC Advanced grant and by a USA-Israeli BSF grant.

†Department of Mathematics, UCLA, Los Angeles, CA, 90095. Email: huanghao@math.ucla.edu.

‡Department of Mathematics, UCLA, Los Angeles, CA 90095. Email: bsudakov@math.ucla.edu. Research supported in part by NSF grant DMS-1101185, NSF CAREER award DMS-0812005 and by USA-Israeli BSF grant.

$n-1, x_2 = \cdots = x_n = -1$ best possible? In other words, when can we guarantee that $A(n,k) = \binom{n-1}{k-1}$? This question was first raised by Bier and Manickam [4, 5] in their study of the so-called first distribution invariant of the Johnson scheme. In 1987, Manickam and Miklós [15] proposed the following conjecture, which in the language of the Johnson scheme was also posed by Manickam and Singhi [16] in 1988.

Conjecture 1.1 *For all $n \geq 4k$, we have $A(n,k) = \binom{n-1}{k-1}$.*

In the Erdős-Ko-Rado theorem, if $n < 2k$, all the k-subsets form an intersecting family of size $\binom{n}{k} > \binom{n-1}{k-1}$. But for $n > 2k$, the star structure, which always takes one fixed element and $k-1$ other arbitrarily chosen elements, will do better than the set of all k-subsets of the first $2k-1$ elements. For a similar reason we have the extra condition $n \geq 4k$ in the Manickam-Mikós-Singhi conjecture. $\binom{n-1}{k-1}$ is not the best construction when n is very small compared to k. For example, take $n = 3k+1$ numbers, 3 of which are equal to $-(3k-2)$ and the other $3k-2$ numbers are 3. It is easy to see that the sum is zero. On the other hand, the nonnegative k-sums are those subsets consisting only of 3's, which gives $\binom{3k-2}{k}$ nonnegative k-sums. It is not difficult to verify that when $k > 2$, $\binom{3k-2}{k} < \binom{(3k+1)-1}{k-1}$. However this kind of construction does not exist for larger n.

The Manickam-Mikós-Singhi conjecture has been open for more than two decades. Only a few partial results of this conjecture are known so far. The most important one among them is that the conjecture holds for all n divisible by k. This claim can be proved directly by considering a random partition of our set of numbers into pairwise disjoint sets, each of size k, but it also follows immediately from Baranyai's partition theorem [2]. This theorem asserts that if $k \mid n$, then the family of all k-subsets of $[n]$ can be partitioned into disjoint subfamilies so that each subfamily is a perfect k-matching. Since the total sum is nonnegative, among the n/k subsets from each subfamily, there must be at least one having a nonnegative sum. Hence there are no less than $\binom{n}{k}/(n/k) = \binom{n-1}{k-1}$ nonnegative k-sums in total. Besides this case, the conjecture is also known to be true for small k. It is not hard to check it for $k = 2$, and the case $k = 3$ was settled by Manickam [14], and by Marino and Chiaselotti [17] independently.

Let $f(k)$ be the minimal number N such that $A(n,k) = \binom{n-1}{k-1}$ for all $n \geq N$. The Manickam-Miklós-Singhi conjecture states that $f(k) \leq 4k$. The existence of such function f was first demonstrated by Manickam and Miklós [15] by showing $f(k) \leq (k-1)(k^k + k^2) + k$. Bhattacharya [3] found a new and shorter proof of existence of f later, but he didn't improve the previous bound. Very recently, Tyomkyn [20] obtained a better upper bound $f(k) \leq k(4e \log k)^k \sim e^{ck \log\log k}$, which is still exponential.

In this paper we discuss a connection between the Manickam-Miklós-Singhi conjecture and a problem about matchings in dense uniform hypergraph. We call a hypergraph H r-uniform if all the edges have size r. Denote by $\nu(H)$ the matching number of H, which is the maximum number of pairwise disjoint edges in H. For our application, we need the fact that if a $(k-1)$-uniform hypergraph on $n-1$ vertices has matching number at most n/k, then its number of edges cannot exceed $c\binom{n-1}{k-1}$ for some constant $c < 1$ independent of n, k. This is closely related to a special case of a long-standing open problem of Erdős [7], who in 1965 asked to determine the maximum possible number of edges of an r-uniform hypergraph H on n vertices with matching number $\nu(H)$. Erdős conjectured that the optimal case is when H is a clique or the complement of a clique, more precisely, for $\nu(H) < \lfloor n/r \rfloor$ the maximum possible number of edges is given by the following equation:

$$\max e(H) = \max \left\{ \binom{r[\nu(H)+1]-1}{r}, \binom{n}{r} - \binom{n-\nu(H)}{r} \right\} \tag{1}$$

For our application to the Manickam-Miklós-Singhi conjecture, it suffices to prove a weaker statement which bounds the number of edges as a function of the fractional matching number $\nu^*(H)$ instead of $\nu(H)$. To attack the latter problem we combine duality with a probabilistic technique together with an inequality by Feige [9] which bounds the probability that the sum of an arbitrary number of nonnegative independent random variables exceeds its expectation by a given amount. Using this machinery, we obtain the first polynomial upper bound $f(k) \le 33k^2$, which substantially improves all the previous exponential estimates.

Theorem 1.2 *Given integers n and k satisfying $n \ge 33k^2$, for any n real numbers $\{x_1, \cdots, x_n\}$ whose sum is nonnegative, there are at least $\binom{n-1}{k-1}$ nonnegative k-sums.*

Recall that earlier we mentioned the similarity between the Manickam-Miklós-Singhi conjecture and the Erdős-Ko-Rado theorem. When $n \ge 4k$, the conjectured extremal example is $x_1 = n-1, x_2 = \cdots = x_n = -1$, where all the $\binom{n-1}{k-1}$ nonnegative k-sums use x_1. For the Erdős-Ko-Rado theorem when $n > 2k$, the extremal family also consists of all the $\binom{n-1}{k-1}$ subsets containing one fixed element. It is a natural question to ask if this kind of structure is forbidden, can we obtain a significant improvement on the $\binom{n-1}{k-1}$ bound? A classical result of Hilton and Milner [12] asserts that if $n > 2k$ and no element is contained in every k-subset, then the intersecting family has size at most $\binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1$, with the extremal example being one of the following two.

- Fix $x \in [n]$ and $X \subset [n] \backslash \{x\}$, $|X| = k$. The family $\mathcal{F}_1$ consists of X and all the k-subsets containing x and intersecting with X.
- Take $Y \subset [n]$, $|Y| = 3$. The family $\mathcal{F}_2$ consists of all the k-subsets of $[n]$ which intersects Y with at least two elements.

It can be easily checked that both families are intersecting and $|\mathcal{F}_1| = \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1$, $|\mathcal{F}_2| = 3\binom{n-3}{k-2} + \binom{n-3}{k-3}$. When $k = 3$, $|\mathcal{F}_1| = |\mathcal{F}_2|$ and their structures are non-isomorphic. For $k \ge 4$, $|\mathcal{F}_1| > |\mathcal{F}_2|$, so only the first construction is optimal.

Here we prove a Hilton-Milner type result about the minimum number of nonnegative k-sums. Call a number x_i *large* if its sum with any other $k-1$ numbers x_j is nonnegative. We prove that if no x_i is large, then the $\binom{n-1}{k-1}$ bound can be greatly improved. We also show that there are two extremal structures, one of which is maximum for every k and the other only for $k = 3$. This result can be considered as an analogue of the two extremal cases mentioned above in the Hilton-Milner theorem.

Theorem 1.3 *For any fixed integer k and sufficiently large n, and for any n real numbers $x_1, \cdots, x_n$ with $\sum_{i=1}^n x_i \ge 0$, where no x_i is large, the number N of different nonnegative k-sums is at least $\binom{n-1}{k-1} + \binom{n-k-1}{k-1} - 1$.*

For large n, Theorem 1.3 (whose statement is tight) improves the $\binom{n-1}{k-1}$ bound in the nonnegative k-sum problem to $\binom{n-1}{k-1}+\binom{n-k-1}{k-1}-1$ when large numbers are forbidden. This bound is asymptotically $(2+o(1))\binom{n-1}{k-1}$.

Call a number x_i $(1-\delta)$-*moderately large*, if there are at least $(1-\delta)\binom{n-1}{k-1}$ nonnegative k-sums using x_i, for some constant $0 \le \delta < 1$. In particular, when $\delta = 0$ this is the definition of a *large* number. If there is no $(1-\delta)$-moderately large number for some positive δ, we can prove a much stronger result asserting that at least a constant proportion of the $\binom{n}{k}$ k-sums are nonnegative. More precisely, we prove the following statement.

Theorem 1.4 *There exists a positive function $g(\delta, k)$, such that for any fixed k and δ and all sufficiently large n, the following holds. For any set of n real numbers $x_1, \cdots, x_n$ with nonnegative sum in which no member is $(1-\delta)$-moderately large, the number N of nonnegative k-sums in the set is at least $g(\delta, k)\binom{n}{k}$.*

The rest of this paper is organized as follows. In the next section we present a quick proof of a slightly worse bound for the function $f(k)$ defined above, namely, we show that $f(k) \le 2k^3$. The proof uses a simple estimate on the number of edges in a hypergraph with a given matching number. The proof of Theorem 1.2 appears in Section 3, where we improve this estimate using more sophisticated probabilistic tools. In Section 4 we prove the Hilton-Milner type results Theorem 1.3 and 1.4. The last section contains some concluding remarks and open problems.

2 Nonnegative k-sums and hypergraph matchings

In this section we discuss the connection of the Manickam-Miklós-Singhi conjecture and hypergraph matchings, and verify the conjecture for $n \ge 2k^3$.

Without loss of generality, we can assume $\sum_{i=1}^n x_i = 0$ and $x_1 \ge x_2 \ge \cdots \ge x_n$ with $x_1 > 0$. If the sum of x_1 and the $k-1$ smallest numbers $x_{n-k+2}, \cdots, x_n$ is nonnegative, there are already $\binom{n-1}{k-1}$ nonnegative k-sums by taking x_1 and any other $k-1$ numbers. Consequently we can further assume that $x_1 + x_{n-k+2} + \cdots + x_n < 0$. As all the numbers sum up to zero, we have

$$x_2 + \cdots + x_{n-k+1} > 0 \tag{2}$$

Let m be the largest integer not exceeding $n-k$ which is divisible by k, then $n-2k+1 \le m \le n-k$. Since the numbers are sorted in descending order, we have

$$x_2 + \cdots + x_{m+1} \ge \frac{m}{n-k}(x_2 + \cdots + x_{n-k+1}) > 0 \tag{3}$$

As mentioned in the introduction, the Manickam-Mikós-Singhi conjecture holds when n is divisible by k by Baranyai's partition theorem, thus there are at least $\binom{m-1}{k-1} \ge \binom{n-2k}{k-1}$ nonnegative k-sums using only numbers from $\{x_2, \cdots, x_{m+1}\}$. From now on we are focusing on counting the number of nonnegative k-sums involving x_1. If this number plus $\binom{n-2k}{k-1}$ is at least $\binom{n-1}{k-1}$, then the Manickam-Miklós-Singhi conjecture is true.

Recall that in the proof of the case $k \mid n$, if we regard all the negative k-sums as edges in a k-uniform hypergraph, then the assumption that all numbers add up to zero provides us the fact that this hypergraph does not have a perfect k-matching. One can prove there are at least $\binom{n-1}{k-1}$ edges in the complement of such a hypergraph, which exactly tells the minimum number of nonnegative k-sums. We utilize the same idea to estimate the number of nonnegative k-sums involving x_1. Construct a $(k-1)$-uniform hypergraph H on the vertex set $\{2, \cdots, n\}$. The edge set $E(H)$ consists of all the $(k-1)$-tuples $\{i_1, \cdots, i_{k-1}\}$ corresponding to the negative k-sum $x_1 + x_{i_1} + \cdots + x_{i_{k-1}} < 0$. Our goal is to show that $e(H) = |E(H)|$ cannot be too large, and therefore there must be lots of nonnegative k-sums involving x_1.

Denote by $\nu(H)$ the matching number of our hypergraph H, which is the maximum number of disjoint edges in H. By definition, every edge corresponds to a $(k-1)$-sum which is less than $-x_1$, thus the sum of the $(k-1)\nu(H)$ numbers corresponding to the vertices in the maximal matching is less than $-\nu(H)x_1$. On the other hand, all the remaining $n-1-(k-1)\nu(H)$ numbers are at most x_1. Therefore $-x_1 = x_2 + \cdots + x_n \le -\nu(H)x_1 + (n-1-(k-1)\nu(H))x_1$. By solving this inequality, we have the following lemma.

Lemma 2.1 *The matching number $\nu(H)$ is at most n/k.*

If the matching number of a hypergraph is known and n is large with respect to k, we are able to bound the number of its edges using the following lemma. We denote by $\overline{H}$ the complement of the hypergraph H.

Lemma 2.2 *If $n > k^3$, any $(k-1)$-uniform hypergraph H on $n-1$ vertices with matching number at most n/k has at least $\frac{1}{k+1}\binom{n-1}{k-1}$ edges missing from it.*

Proof. Consider a random permutation σ on the $n-1$ vertices $v_1, \cdots, v_{n-1}$ of H. Let the random variables $Z_1 = 1$ if $(\sigma(v_1), \cdots, \sigma(v_{k-1}))$ is an edge in H and 0 otherwise. Repeat the same process for the next $k-1$ indices and so on. Finally we will have at least $m \ge \frac{n-k}{k-1}$ random variables $Z_1, \cdots, Z_m$. Let $Z = Z_1 + \cdots + Z_m$. Z is always at most n/k since there is no matching of size larger than n/k. On the other hand, $\mathbb{E}Z_i$ is the probability that $k-1$ randomly chosen vertices form an edge in H, therefore $\mathbb{E}Z_i = e(H)/\binom{n-1}{k-1}$. Hence,

$$\frac{n}{k} \ge \mathbb{E}Z = m\frac{e(H)}{\binom{n-1}{k-1}} \ge \frac{n-k}{k-1}\frac{e(H)}{\binom{n-1}{k-1}} \tag{4}$$

The number of edges missing is equal to $e(\overline{H}) = \binom{n-1}{k-1} - e(H)$. By (4), $e(H) \le (1-\frac{1}{k})\frac{n}{n-k}\binom{n-1}{k-1}$, therefore

$$\begin{aligned} e(\overline{H}) &\ge \left[1 - \left(1-\frac{1}{k}\right)\frac{n}{n-k}\right]\binom{n-1}{k-1} \\ &\ge \left[1 - \left(1-\frac{1}{k}\right)\frac{k^3}{k^3-k}\right]\binom{n-1}{k-1} \\ &= \frac{1}{k+1}\binom{n-1}{k-1} \end{aligned} \tag{5}$$

□

Now we can easily prove a polynomial upper bound for the function $f(k)$ considered in the introduction, showing that $f(k) \le 2k^3$.

Theorem 2.3 *If $n \geq 2k^3$, then for any n real numbers $\{x_1, \cdots, x_n\}$ whose sum is nonnegative, the number of nonnegative k-sums is at least $\binom{n-1}{k-1}$.*

Proof. By Lemma 2.2, there are at least $\frac{1}{k+1}\binom{n-1}{k-1}$ edges missing in H, which also gives a lower bound for the number of nonnegative k-sums involving x_1. Together with the previous $\binom{n-2k}{k-1}$ nonnegative k-sums without using x_1, there are at least $\frac{1}{k+1}\binom{n-1}{k-1} + \binom{n-2k}{k-1}$ nonnegative k-sums in total. We claim that this number is greater than $\binom{n-1}{k-1}$ when $n \geq 2k^3$. This statement is equivalent to proving $\binom{n-2k}{k-1}/\binom{n-1}{k-1} \geq 1 - 1/(k+1)$, which can be completed as follows:

$$\begin{aligned}
\binom{n-2k}{k-1}\Big/\binom{n-1}{k-1} &= \left(1-\frac{2k-1}{n-1}\right)\left(1-\frac{2k-1}{n-2}\right)\cdots\left(1-\frac{2k-1}{n-k+1}\right) \\
&\geq 1-\frac{(2k-1)(k-1)}{n-k+1} \\
&\geq 1-\frac{(2k-1)(k-1)}{2k^3-k+1} \\
&\geq 1-\frac{1}{k+1} \qquad (6)
\end{aligned}$$

The last inequality is because $(2k-1)(k-1)(k+1) = 2k^3 - k^2 - 2k + 1 \leq 2k^3 - k + 1$. □

3 Fractional covers and small deviations

The method above verifies the Manickam-Miklós-Singhi conjecture for $n \geq 2k^3$ and improves the current best exponential lower bound $n \geq k(4e \log k)^k$ by Tyomkyn [20]. However if we look at Lemma 2.2 attentively, there is still some room to improve it. Recall our discussion of Erdős' conjecture in the introduction: if the conjecture is true in general, then in order to minimize the number of edges in a $(k-1)$-hypergraph of a given matching number $\nu(H) = n/k$, the hypergraph must be either a clique of size $(k-1)(n/k+1)-1$ or the complement of a clique of size $n - n/k$.

$$e(H) \sim \min\left\{\binom{(1-1/k)n}{k-1}, \binom{n-1}{k-1} - \binom{n-n/k}{k-1}\right\} \sim (1-\frac{1}{k})^{k-1}\binom{n-1}{k-1} \leq \frac{1}{2}\binom{n-1}{k-1} \qquad (7)$$

In this case, the number of edges missing from H must be at least $\frac{1}{2}\binom{n-1}{k-1}$, which is far larger than the bound $\frac{1}{k+1}\binom{n-1}{k-1}$ in Lemma 2.2. If in our proof of Theorem 2.3, the coefficient before $\binom{n-1}{k-1}$ can be changed to a constant instead of the original $\frac{1}{k+1}$, the theorem can also be sharpened to $n \geq \Omega(k^2)$. Based on this idea, in the rest of this section we are going to prove Lemma 3.3, which asserts that $e(H) \geq c\binom{n-1}{k-1}$ for some constant c independent of n and k, and can be regarded as a strengthening of Lemma 2.2. Then we use it to prove our main result of this paper, Theorem 3.5. In order to improve Lemma 2.2, instead of using the usual matching number $\nu(H)$, it suffices to consider its fractional relaxation, which is defined as follows.

$$\begin{aligned}
\nu^*(H) = \max \sum_{e\in E(H)} w(e) &\qquad w: E(H) \to [0,1] \\
\text{subject to } \sum_{i\in e} w(e) \leq 1 &\quad \text{for every vertex } i.
\end{aligned} \qquad (8)$$

Note that $\nu^*(H)$ is always greater or equal than $\nu(H)$. On the other hand, for our hypergraph we can prove the same upper bound for the fractional matching number $\nu^*(H)$ as in Lemma 2.1. Recall that H is the $(k-1)$-uniform hypergraph on the $n-1$ vertices $\{2, \cdots, n\}$, whose edges are those $(k-1)$-tuples $(i_1, \cdots, i_{k-1})$ corresponding to negative k-sums $x_1 + x_{i_1} + \cdots + x_{i_k} < 0$.

Lemma 3.1 *The fractional matching number $\nu^*(H)$ is at most n/k.*

Proof. Choose a weight function $w : E(H) \to [0,1]$ which optimizes the linear program (8) and gives the fractional matching number $\nu^*(H)$, then $\nu^*(H) = \sum_{e \in E(H)} w(e)$. Two observations can be easily made: (i) if $e \in E(H)$, then $\sum_{i \in e} x_i < -x_1$; (ii) $x_i \leq x_1$ for any $i = 2, \cdots, n$ since $\{x_i\}$ are in descending order. Therefore we can bound the fractional matching number in a few steps.

$$\begin{aligned}
x_1 + x_2 + \cdots + x_n &= x_1 + \sum_{i=2}^{n} \Big(\sum_{i \in e} w(e) \Big) x_i + \sum_{i=2}^{n} \Big(1 - \sum_{i \in e} w(e) \Big) x_i \\
&\leq x_1 + \sum_{e \in E(H)} \Big(\sum_{i \in e} x_i \Big) w(e) + \sum_{i=2}^{n} \Big(1 - \sum_{i \in e} w(e) \Big) x_1 \\
&\leq x_1 + \sum_{e \in E(H)} w(e)(-x_1) + \Big(n - 1 - \sum_{e \in E(H)} \sum_{i \in e} w(e) \Big) x_1 \\
&= x_1 - \nu^*(H) x_1 + \Big(n - 1 - (k-1)\nu^*(H) \Big) x_1 \\
&= (n - k\nu^*(H)) x_1 \qquad (9)
\end{aligned}$$

Lemma 3.1 follows from this inequality and our assumption that $x_1 + \cdots + x_n = 0$ and $x_1 > 0$. □

The determination of the fractional matching number is actually a linear programming problem. Therefore we can consider its dual problem, which gives the fractional covering number $\tau^*(H)$.

$$\begin{aligned}
\tau^*(H) = \min \sum_i v(i) \qquad & v : V(H) \to [0,1] \\
\text{subject to } \sum_{i \in e} v(i) \geq 1 \quad & \text{for every edge } e.
\end{aligned} \qquad (10)$$

By duality we have $\tau^*(H) = \nu^*(H) \leq n/k$. Getting a upper bound for $e(H)$ is equivalent to finding a function $v : V(H) \to [0,1]$ satisfying $\sum_{i \in V(H)} v(i) \leq n/k$ that maximizes the number of $(k-1)$-tuples e where $\sum_{i \in e} v(i) \geq 1$. Since this number is monotone increasing in every $v(i)$, we can assume that it is maximized by a function v with $\sum_{i \in V(H)} v(i) = n/k$.

The following lemma was established by Feige [9], and later improved by He, Zhang, and Zhang [11]. It bounds the tail probability of the sum of independent nonnegative random variables with given expectations. It is stronger than Markov's inequality in the sense that the number of variables m does not appear in the bound.

Lemma 3.2 *Given m independent nonnegative random variables $X_1, \cdots, X_m$, each of expectation at most 1, then*

$$Pr\Big(\sum_{i=1}^{m} X_i < m + \delta \Big) \geq \min \Big\{ \delta/(1+\delta), \frac{1}{13} \Big\} \qquad (11)$$

Now we can show that the complement of the hypergraph H has at least constant edge density, which implies as a corollary that a constant proportion of the k-sums involving x_1 must be nonnegative.

Lemma 3.3 *If $n \geq Ck^2$ with $C \geq 1$, and H is a $(k-1)$-uniform hypergraph on $n-1$ vertices with fractional covering number $\tau^*(H) = n/k$, then there are at least $\left(\frac{1}{13} - \frac{1}{2C}\right)\frac{(n-1)^{k-1}}{(k-1)!}$ $(k-1)$-sets which are not edges in H.*

Proof. Choose a weight function $v : V(H) \to [0,1]$ which optimizes the linear programming problem (10). Define a sequence of $k-1$ independent and identically distributed random variables $X_1, \cdots, X_{k-1}$, such that for any $1 \leq j \leq k-1, 2 \leq i \leq n$, $X_j = v(i)$ with probability $1/(n-1)$. It is easy to compute the expectation of X_i, which is

$$\mathbb{E}X_i = \frac{1}{n-1}\sum_{i=2}^{n} v(i) = \frac{n}{k(n-1)} \tag{12}$$

Now we can estimate the number of $(k-1)$-tuples with sum less than 1. The probability of the event $\left\{\sum_{i=1}^{k-1} X_i < 1\right\}$ is basically the same as the probability that a random $(k-1)$-tuple has sum less than 1, except that two random variables X_i and X_j might share a weight from the same vertex, which is not allowed for forming an edge. However, we assumed that n is much larger than k, so this error term is indeed negligible for our application. Note that for $i_1 < \cdots < i_{k-1}$, the probability that $X_j = v(i_j)$ for every $1 \leq j \leq k-1$ is equal to $1/(n-1)^{k-1}$.

$$\begin{aligned}
e(\overline{H}) &= \left|\left\{i_1 < \cdots < i_{k-1} : v(i_1) + \cdots + v(i_{k-1}) < 1\right\}\right| \\
&= \frac{(n-1)^{k-1}}{(k-1)!} \sum_{\text{distinct } i_1,\cdots,i_{k-1}} Pr\Big[X_1 = v(i_1), \cdots, X_{k-1} = v(i_{k-1}); \sum_{i=1}^{k-1} X_i < 1\Big] \\
&\geq \frac{(n-1)^{k-1}}{(k-1)!}\Big[Pr\Big(\sum_{i=1}^{k-1} X_i < 1\Big) - \sum_{l}\sum_{i\neq j} Pr\Big(X_i = X_j = v(i_l)\Big)\Big] \\
&\geq \frac{(n-1)^{k-1}}{(k-1)!}\Big[Pr\Big(\sum_{i=1}^{k-1} X_i < 1\Big) - \frac{\binom{k-1}{2}}{n-1}\Big] \\
&\geq \frac{(n-1)^{k-1}}{(k-1)!}\Big[Pr\Big(\sum_{i=1}^{k-1} X_i < 1\Big) - \frac{1}{2C}\Big]
\end{aligned} \tag{13}$$

The last inequality is because $n \geq Ck^2$ and $-1 \geq -3Ck + 2C$ for $C \geq 1, k \geq 1$, and the sum of these two inequalities implies that $\frac{(k-1)(k-2)}{2(n-1)} \leq \frac{1}{2C}$.

Define $Y_i = X_i \cdot k(n-1)/n$ to normalize the expectations to $\mathbb{E}Y_i = 1$. Y_i's are nonnegative because each vertex receives a nonnegative weight in the linear program (10). Applying Lemma 3.2

to $Y_1, \cdots, Y_{k-1}$ and setting $m = k-1$, $\delta = (n-k)/n$, we get

$$\begin{aligned} Pr\Big(X_1 + \cdots + X_{k-1} < 1\Big) &= Pr\Big(Y_1 + \cdots + Y_{k-1} < k(n-1)/n\Big) \\ &\geq \min\Big\{\frac{n-k}{2n-k}, \frac{1}{13}\Big\} \end{aligned} \tag{14}$$

When $n > Ck^2$ and $k \geq 2$, $C \geq 1$, we have

$$\frac{n-k}{2n-k} > \frac{Ck^2-k}{2Ck^2-k} = \frac{Ck-1}{2Ck-1} \geq \frac{1}{13} \tag{15}$$

Combining (13) and (14) we immediately obtain Lemma 3.3. □

Corollary 3.4 *If $n \geq Ck^2$ with $C \geq 1$, then there are at least $\Big(\frac{1}{13} - \frac{1}{2C}\Big)\frac{(n-1)^{k-1}}{(k-1)!}$ nonnegative k-sums involving x_1.*

Now we are ready to prove our main theorem:

Theorem 3.5 *If $n \geq 33k^2$, then for any n real numbers $x_1, \cdots, x_n$ with $\sum_{i=1}^n x_i \geq 0$, the number of nonnegative k-sums is at least $\binom{n-1}{k-1}$.*

Proof. By the previous discussion, we know that there are at least $\binom{n-2k}{k-1}$ nonnegative k-sums using only $x_2, \cdots, x_n$. By Corollary 3.4, there are at least $\Big(\frac{1}{13} - \frac{1}{2\cdot 33}\Big)\frac{(n-1)^{k-1}}{(k-1)!} \geq \frac{2}{33}\frac{(n-1)^{k-1}}{(k-1)!}$ nonnegative k-sums involving x_1. In order to prove the theorem, we only need to show that for $n \geq 33k^2$,

$$\frac{2}{33}\frac{(n-1)^{k-1}}{(k-1)!} + \binom{n-2k}{k-1} \geq \binom{n-1}{k-1} \tag{16}$$

Define an infinitely differentiable function $g(x) = \binom{x}{k-1} = \frac{x(x-1)\cdots(x-k+2)}{(k-1)!}$. It is not difficult to see $g''(x) > 0$ when $x > k-1$. Therefore

$$\binom{n-1}{k-1} - \binom{n-2k}{k-1} = g(n-1) - g(n-2k) \leq [(n-1)-(n-2k)]g'(n-1) = (2k-1)g'(n-1) \tag{17}$$

$$\begin{aligned} g'(x) &= \frac{(x-1)(x-2)\cdots(x-k+2)}{(k-1)!} + \frac{x(x-2)\cdots(x-k+2)}{(k-1)!} + \cdots + \frac{x(x-1)\cdots(x-k+3)}{(k-1)!} \\ &\leq (k-1)\frac{x(x-1)\cdots(x-k+3)}{(k-1)!} \\ &\leq (k-1)\frac{x^{k-2}}{(k-1)!} \end{aligned} \tag{18}$$

Combining (17) and (18),

$$\binom{n-1}{k-1} - \binom{n-2k}{k-1} \leq (2k-1)g'(n-1) \leq (2k-1)(k-1)\frac{(n-1)^{k-2}}{(k-1)!} \leq \frac{2}{33}\frac{(n-1)^{k-1}}{(k-1)!} \tag{19}$$

The last inequality follows from our assumption $n \geq 33k^2$. □

4 Hilton-Milner type results

In this section we prove two Hilton-Milner type results about the minimum number of nonnegative k-sums. The first theorem asserts that for sufficiently large n, if $\sum_{i=1}^{n} x_i \geq 0$ and no x_i is large, then there are at least $\binom{n-1}{k-1} + \binom{n-k-1}{k-1} - 1$ nonnegative k-sums.

Proof of Theorem 1.3. We again assume that $x_1 \geq \cdots \geq x_n$ and $\sum_{i=1}^{n} x_i$ is zero. Since x_1 is not large, we know that there exists a $(k-1)$-subset S_1 not containing 1, such that $x_1 + \sum_{i \in S_1} x_i < 0$. Suppose t is the largest integer so that there are t subsets $S_1, \cdots, S_t$, such that for any $1 \leq j \leq t$, S_j is disjoint from $\{1, \cdots, j\}$, has size at most $j(k-1)$ and

$$x_1 + \cdots + x_j + \sum_{i \in S_j} x_i < 0.$$

As we explained above $t \geq 1$ and since $x_1 \geq \cdots \geq x_n$ we may also assume that S_j consists of the last $|S_j|$ indices in $\{1, \cdots, n\}$. By Corollary 3.4, for sufficiently large n, there are at least $\frac{1}{14}\binom{n-1}{k-1}$ nonnegative k-sums using x_1. Note also that after deleting x_1 and $\{x_i\}_{i \in S_1}$, the sum of the remaining $n - 1 - |S_1| \geq n - k$ numbers is nonnegative. Therefore, again by Corollary 3.4, there are at least $\frac{1}{14}\binom{n-k-1}{k-1}$ nonnegative k-sums using x_2 but not x_1. In the next step, we delete x_1, x_2 and $\{x_i\}_{i \in S_2}$ and bound the number of nonnegative k-sums involving x_3 but neither x_1 or x_2 by $\frac{1}{14}\binom{n-2k-1}{k-1}$. Repeating this process for 30 steps, we obtain

$$\begin{aligned} N &\geq \frac{1}{14}\left[\binom{n-1}{k-1} + \binom{n-k-1}{k-1} + \cdots + \binom{n-29k-1}{k-1}\right] \\ &> \frac{30}{14}\binom{n-29k-1}{k-1} > 2\binom{n-1}{k-1} > \binom{n-1}{k-1} + \binom{n-k-1}{k-1} - 1 \end{aligned}$$

where here we used the fact that $\frac{30}{14} > 2$ and n is sufficiently large (as a function of k).

If $2 \leq t < 30$, by the maximality of t, we know that the sum of x_{t+1} with any $(k-1)$ numbers with indices not in $\{1, \cdots, t+1\} \cup S_t$ is nonnegative. This gives us $\binom{n-(t+1)-|S_t|}{k-1} \geq \binom{n-tk-1}{k-1}$ nonnegative k-sums. We can also replace x_{t+1} by any x_i where $1 \leq i \leq t$ and the new k-sum is still nonnegative since $x_i \geq x_{t+1}$. Therefore,

$$N \geq (t+1)\binom{n-tk-1}{k-1} \geq (t+1)\binom{n-29k-1}{k-1} > 2\binom{n-1}{k-1}$$

for sufficiently large n. Thus the only remaining case is $t = 1$.

Recall that x_1 is not large, and hence $x_1 + (x_{n-k+2} + \cdots + x_n) < 0$. Suppose I is a $(k-1)$-subset of $[2, n]$ such that $x_1 + \sum_{i \in I} x_i < 0$. If $2 \in I$, then $x_1 + x_2 + \sum_{i \in I \backslash \{2\}} x_i < 0$, this contradicts the assumption $t = 1$ since $|I \backslash \{2\}| = k - 2 \leq 2(k-1)$. Hence we can assume that all the $(k-1)$-subsets $I_1, \cdots, I_m$ corresponding to negative k-sums involving x_1 belong to the interval $[3, n]$. Let N_1 be the number of nonnegative k-sums involving x_1, and let N_2 be the number of nonnegative k-sums using x_2 but not x_1, then

$$N \geq N_1 + N_2 = \left[\binom{n-1}{k-1} - m\right] + N_2.$$

In order to prove $N \geq \binom{n-1}{k-1} + \binom{n-k-1}{k-1} - 1$, we only need to establish the following inequality

$$N_2 \geq \binom{n-k-1}{k-1} + m - 1. \tag{20}$$

Observe that the subsets $I_1, \cdots, I_m$ satisfy some additional properties. First of all, if two sets I_i and I_j are disjoint, then by definition, $x_1 + \sum_{l \in I_i} x_l < 0$ and $x_2 + \sum_{l \in I_j} x_l \leq x_1 + \sum_{l \in I_j} x_l < 0$, summing them up gives $x_1 + x_2 + \sum_{l \in I_i \cup I_j} x_l < 0$ with $|I_i \cup I_j| = 2(k-1)$, which again contradicts the assumption $t = 1$. Therefore we might assume that $\{I_i\}_{1 \leq i \leq m}$ is an intersecting family. By the Erdős-Ko-Rado theorem,

$$m \leq \binom{(n-2)-1}{(k-1)-1} = \binom{n-3}{k-2}.$$

The second observation is that if a $(k-1)$-subset $I \subset [3, n]$ is disjoint from some I_i, then $x_2 + \sum_{i \in I} x_i \geq 0$. Otherwise if $x_2 + \sum_{i \in I} x_i < 0$ and $x_1 + \sum_{k \in I_i} x_k < 0$, for the same reason this contradicts $t = 1$. Hence N_2 is bounded from below by the number of $(k-1)$-subsets $I \subset [3, n]$ such that I is disjoint from at least one of $I_1, \cdots, I_m$. Equivalently we need to count the distinct $(k-1)$-subsets contained in some $J_i = [3, n] \backslash I_i$, all of which have sizes $n-k-1$. By the real version of the Kruskal-Katona theorem (see Ex.13.31(b) in [13]), if $m = \binom{x}{n-k-1}$ for some positive real number $x \geq n-k-1$, then $N_2 \geq \binom{x}{k-1}$. On the other hand, it is already known that $1 \leq m \leq \binom{n-3}{k-2} = \binom{n-3}{n-k-1}$, thus $n-k-1 \leq x \leq n-3$. The only remaining step is to verify the following inequality for x in this range,

$$\binom{x}{k-1} \geq \binom{n-k-1}{k-1} + \binom{x}{n-k-1} - 1. \tag{21}$$

Let $f(x) = \binom{x}{k-1} - \binom{x}{n-k-1}$, note that when $x \leq n-4 = (k-2) + (n-k-2)$,

$$\begin{aligned} f(x+1) - f(x) &= \left[\binom{x+1}{k-1} - \binom{x+1}{n-k-1}\right] - \left[\binom{x}{k-1} - \binom{x}{n-k-1}\right] \\ &= \binom{x}{k-2} - \binom{x}{n-k-2} \geq 0 \end{aligned}$$

The last inequality is because when n is large, $x \geq n-k-1 > 2(k-2)$. Moreover, $\binom{x}{t}$ is an increasing function for $0 < t < x/2$, so when $x \leq n-4$, $\binom{x}{n-k-2} = \binom{x}{x-(n-k-2)} \leq \binom{x}{k-2}$.

Therefore we only need to verify (21) for $n-k-1 \leq x < n-k$, which corresponds to $1 \leq m \leq n-k-1$. For $m = 1$, (21) is obvious, so it suffices to look at the case $m \geq 2$. The number of distinct $(k-1)$-subsets of J_1 or J_2 is minimized when $|J_1 \cap J_2| = n-k-2$, which, by the inclusion-exclusion principle, gives

$$N_2 \geq 2\binom{n-k-1}{k-1} - \binom{n-k-2}{k-1} = \binom{n-k-1}{k-1} + \binom{n-k-2}{k-2}.$$

So (20) is also true for $2 \leq m \leq \binom{n-k-2}{k-2} + 1$. It is easy to see that for $k \geq 3$ and n sufficiently large, $n-k-1 \leq \binom{n-k-2}{k-2} + 1$. For $k = 2$, we have $x = n-3$ and (21) becomes $\binom{n-3}{1} \geq \binom{n-3}{1} + \binom{n-3}{n-3} - 1$, which is true and completes the proof. □

Remark 1. In order for all the inequalities to be correct, we only need $n > Ck^2$. By carefully analyzing the above computations, one can check that $C = 500$ is enough.

Remark 2. Note that in the proof, the equality (20) holds in two different cases. The first case is when $m = 1$, which means $x_1 + x_{n-k+2} + \cdots + x_n < 0$ but any other k-sums involving x_1 are nonnegative. All the other nonnegative k-sums are formed by x_2 together with any $(k-1)$-subsets not containing $x_{n-k+2}, \cdots, x_n$. This case is realizable by the following construction: $x_1 = k(k-1)n$, $x_2 = n-2$, $x_3 = \cdots = x_{n-k+1} = -1$, $x_{n-k+2} = \cdots = x_n = -(kn+1)$. The second case is in (21) when $x = n-4$ and $x = n-k-1$ holds simultaneously, which gives $k = 3$. In this case, $m = \binom{n-3}{n-4} = n-3$, and the Kruskal-Katona theorem holds with equality for the $(n-4)$-subsets $J_1, \cdots, J_{n-3}$. That is to say, the negative 3-sums using x_1 are $x_1 + x_i + x_n$ for $3 \le i \le n-1$, while the nonnegative 3-sums containing x_2 but not x_1 are $x_2 + x_i + x_j$ for $3 \le i < j \le n-1$. This case can also be achieved by setting $x_1 = x_2 = 1$, $x_3 = \cdots = x_{n-1} = \frac{1}{2(n-3)}$, and $x_n = -\frac{3}{2}$. For large n, these are the only two possible configurations achieving equality in Theorem 1.3.

Next we prove Theorem 1.4, which states that if $\sum_i x_i \ge 0$ and no x_i is moderately large, then at least a constant proportion of the $\binom{n}{k}$ k-sums are nonnegative.

Proof of Theorem 1.4. Suppose t is the largest integer so that there are t subsets $S_1, \cdots, S_t$ such that for any $1 \le j \le t$, S_j is disjoint from $\{1, \cdots, j\}$, has at most $j(k-1)$ elements, and

$$x_1 + \cdots + x_j + \sum_{i \in S_j} x_i < 0.$$

By the maximality of t, the sum of x_{t+1} and any $k-1$ numbers x_i with indices from $[n]\backslash(\{1, \cdots, t+1\} \cup S_t)$ is nonnegative, so there are at least $\binom{n-tk-1}{k-1}$ nonnegative k-sums using x_{t+1}. Since x_{t+1} is not $(1-\delta)$-moderately large,

$$\binom{n-tk-1}{k-1} < (1-\delta)\binom{n-1}{k-1}.$$

For sufficiently large n, this is asymptotically equivalent to

$$\left(1 - \frac{tk}{n}\right)^{k-1} < 1 - \delta.$$

Since

$$\left(1 - \frac{tk}{n}\right)^{k-1} > 1 - \frac{tk(k-1)}{n},$$

we have

$$t > \frac{n}{k^2}\delta$$

Recall that by Corollary 3.4, for each $i = 1, \cdots, \frac{n}{k^2}\delta$, x_i gives at least $\frac{1}{14}\binom{n-(i-1)k-1}{k-1}$ nonnegative k-sums, therefore

$$\begin{aligned} N &\ge \frac{1}{14}\left[\binom{n-1}{k-1} + \cdots + \binom{n-(\frac{n}{k^2}\delta)k-1}{k-1}\right] \\ &\ge \frac{n\delta}{14k^2}\binom{n-(\frac{n}{k^2}\delta)k-1}{k-1} \\ &= \frac{\delta}{14k}\binom{n}{k}\left(1 - \frac{\delta n/k}{n-1}\right)\cdots\left(1 - \frac{\delta n/k}{n-k+1}\right) \\ &\ge \frac{\delta}{14k}\binom{n}{k}\left(1 - \frac{\delta n}{n-k+1}\right) \end{aligned}$$

Since $\delta < 1$, when $n \geq \dfrac{k-1}{1-\sqrt{\delta}}$, we have $\dfrac{\delta n}{n-k+1} \leq \sqrt{\delta}$. Therefore setting $g(\delta, k) = \dfrac{\delta(1-\sqrt{\delta})}{14k}$ completes the proof. □

5 Concluding remarks

- In this paper, we have proved that if $n > 33k^2$, any n real numbers with a nonnegative sum have at least $\binom{n-1}{k-1}$ nonnegative k-sums, thereby verifying the Manickam-Miklós-Singhi conjecture in this range. Because of the inequality $\binom{n-2k}{k} + C\binom{n-1}{k-1} \geq \binom{n-1}{k-1}$ we used, our method will not give a better range than the quadratic one, and we did not try hard to compute the best constant in the quadratic bound. It would be interesting to decide if the Manickam-Miklós-Singhi conjecture can be verified for a linear range $n > ck$. Perhaps some algebraic methods or structural analysis of the extremal configurations will help.

- Feige [9] conjectures that the constant $1/13$ in Lemma 3.2 can be improved to $1/e$. This is a special case of a more general question suggested by Samuels [18]. He asked to determine, for a fixed m, the infimum of $Pr(X_1 + \cdots + X_k < m)$, where the infimum is taken over all possible collections of k independent nonnegative random variables $X_1, \cdots, X_k$ with given expectations $\mu_1, \cdots, \mu_k$. For $k = 1$ the answer is given by Markov's inequality. Samuels [18, 19] solved this question for $k \leq 4$, but for all $k \geq 5$ his problem is still completely open.

- Another intriguing objective is to prove the conjecture by Erdős which states that the maximum number of edges in an r-uniform hypergraph H on n vertices with matching number $\nu(H)$ is exactly

$$\max\left\{\binom{r[\nu(H)+1]-1}{r}, \binom{n}{r} - \binom{n-\nu(H)}{r}\right\}.$$

  The first number corresponds to a clique and the second case is the complement of a clique. When $\nu(H) = 1$, this conjecture is exactly the Erdős-Ko-Rado theorem [8]. Erdős also verified it for $n > c_r\nu(H)$ where c_r is a constant depending on r. Recall that in our graph H we have $\nu(H) \leq n/k$ and r here is equal to $k-1$, so if Erdős' conjecture is true in general, we can give a direct proof of constant edge density in the complement of H. In this way we can avoid using fractional matchings in our proof. But even without the application here, this conjecture is interesting in its own right. The fractional version of Erdős' conjecture is also very interesting. In its asymptotic form it says that if H is an r-uniform n-vertex hypergraph with fractional matching number $\nu^*(H) = xn$, where $0 \leq x < 1/r$, then

$$e(H) \leq (1+o(1)) \max\left\{(rx)^r, 1-(1-x)^r\right\}\binom{n}{r}. \tag{22}$$

- As pointed out to us by Andrzej Ruciński, part of our reasoning in Section 3 implies that the function $A(n,k)$ defined in the first page is precisely $\binom{n}{k}$ minus the maximum possible number of edges in a k-uniform hypergraph on n vertices with fractional covering number strictly smaller than n/k. Indeed, given n reals $x_1, \ldots, x_n$ with sum zero and only $A(n,k)$ nonnegative k-sums, we may assume that the absolute value of each x_i is smaller than $1/k$ (otherwise simply multiply all of them by a sufficiently small positive real.) Next, add a sufficiently small positive ϵ to each x_i, keeping each x_i smaller than $1/k$ and keeping the sum of any negative k-tuple below zero (this is clearly possible.) Note that the sum of these new reals, call them x_i', is strictly positive and the number of positive k-sums is $A(n,k)$. Put $\nu(i) = 1/k - x_i'$ and observe that $\sum_i \nu(i) < n/k$ and the k-uniform hypergraph whose edges are all k-sets e for which $\sum_{i \in e} \nu(i) \geq 1$ has exactly $\binom{n}{k} - A(n,k)$ edges. Therefore, there is a k-uniform hypergraph on n vertices with fractional covering number strictly smaller than n/k and at least $\binom{n}{k} - A(n,k)$ edges. Conversely, given a k-uniform hypergraph H on n vertices and a fractional covering of it $\nu : V(H) \mapsto [0,1]$ with $\sum_i \nu(i) = n/k - \delta < n/k$ and $\sum_{i \in e} \nu(i) \geq 1$ for each $e \in E(H)$, one can define $x_i = \frac{1}{k} - \frac{\delta}{n} - \nu(i)$ to get a set of n reals whose sum is zero, in which the number of nonnegative k-sums is at most $\binom{n}{k} - |E(H)|$ (as the sum of the numbers x_i for every k-set forming an edge of H is at most $1 - \frac{k\delta}{n} - 1 < 0$). This implies the desired equality, showing that the problem of determining $A(n,k)$ is equivalent to that of finding the maximum possible number of edges of a k-uniform hypergraph on n vertices with fractional covering number strictly smaller than n/k. Note that this is equivalent to the problem of settling the fractional version of the conjecture of Erdős for the extremal case of fractional matching number $< n/k$.

- Although the fractional version of Erdős' conjecture is still widely open in general, our techniques can be used to make some progress on this problem. Combining the approach from Section 3 and the above mentioned results of Samuels we verified, in joint work with Frankl, Rödl and Ruciński [1], conjecture (22) for certain ranges of x for 3 and 4-uniform hypergraphs. These results can be used to study a Dirac-type question of Daykin and Häggkvist [6] and of Hán, Person and Schacht [10] about perfect matchings in hypergraphs.

  For an r-uniform hypergraph H and for $1 \leq d \leq r$, let $\delta_d(H)$ denote the minimum number of edges containing a subset of d vertices of H, where the minimum is taken over all such subsets. In particular, $\delta_1(H)$ is the minimum vertex degree of H. For r that divides n, let $m_d(r,n)$ denote the smallest number, so that any r-uniform hypergraph H on n vertices with $\delta_d(H) \geq m_d(r,n)$ contains a perfect matching. Similarly, let $m_d^*(r,n)$ denote the smallest number, so that any r-uniform hypergraph H on n vertices with $\delta_d(H) \geq m_d^*(r,n)$ contains a perfect fractional matching.

  Together with Frankl, Rödl and Ruciński [1] we proved that for all d and r, $m_d(r,n) \sim m_d^*(r,n)$ and further reduced the problem of determining the asymptotic behavior of these numbers to some special cases of conjecture (22). Using this relation we were able to determine $m_d(r,n)$ asymptotically for several values of d and r, which have not been known before. Moreover, our approach may lead to a solution of the general case as well, see [1] for the details.

Acknowledgment We would like to thank Andrzej Ruciński for helpful discussions and comments, and Nati Linial for inspiring conversations and intriguing questions which led us to the results in Section 4.

References

[1] N. Alon, P. Frankl, H. Huang, V. Rödl, A. Ruciński and B. Sudakov, Large matchings in uniform hypergraphs and the conjectures of Erdos and Samuels, submitted.

[2] Z. Baranyai, On the factorization of the complete uniform hypergraph, *Colloq. Math. Soc. János Bolyai* **10** (1975), 91–108.

[3] A. Bhattacharya, On a conjecture of Manickam and Singhi, *Discrete Math.* **272** (2003), 259–261

[4] T. Bier, A distribution invariant for the association schemes and strongly regular graphs, *Linear algebra and its applications* **57** (1984), 230–2521.

[5] T. Bier and N. Manickam, The first distribution invariant of the Johnson scheme, *SEAMS Bull. Math.* **11** (1987), 61–68.

[6] D. E. Daykin and R. Häggkvist, Degrees giving independent edges in a hypergraph, *Bull. Austral. Math. Soc.* **23** (1981), 103–109.

[7] P. Erdős, A problem on independent r-tuples, *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.* **8** (1965), 93–95.

[8] P. Erdős, C. Ko and R. Rado, Intersection theorems for systems of finite sets, *Quart. J. Math. Oxford. Series (2)* **12** (1961), 312–320.

[9] U. Feige, On sums of independent random variables with unbounded variance and estimating the average degree in a graph, *SIAM J. Comput.* **35** (2006), no. 4, 964–984.

[10] H. Hán, Y. Person and M. Schacht, On perfect matchings in uniform hypergraphs with large minimum vertex degree, *SIAM J. Discrete Math.* **23** (2009), no. 2, 732–748.

[11] S. He, J. Zhang and S. Zhang, Bounding probability of small deviation: a fourth moment approach, *Math. Oper. Res.* **35** (2010), no.1, 208–232.

[12] A. J. W. Hilton and E. C. Milner, Some intersection theorems for systems of finite sets, *Quart. J. Math. Oxford Ser. (2)*, **18** (1967), 369–384.

[13] L. Lovász, *Combinatorial problems and exercises*, North-Holland Publishing Co., Amsterdam, 1979.

[14] N. Manickam, On the distribution invariants of association schemes, Ph.D. Dissertation, Ohio State University, 1986.

[15] N. Manickam and D. Miklós, On the number of non-negative partial sums of a non-negative sum, *Colloq. Math. Soc. Janos Bolyai* **52** (1987), 385–392.

[16] N. Manickam and N. M. Singhi, First distribution invariants and EKR theorems, *J. Combinatorial Theory, Series A* **48** (1988), 91–103.

[17] G. Marino and G. Chiaselotti, A method to count the positive 3-subsets in a set of real numbers with non-negative sum, *European J. Combin.* **23** (2002), 619–629.

[18] S.M. Samuels, On a Chebyshev-type inequality for sums of independent random variables, *Ann. Math. Statist.* **37** (1966), 248–259.

[19] S.M. Samuels, More on a Chebyshev-type inequality for sums of independent random variables, Purdue Stat. Dept. Mimeo. Series no. 155 (1968)

[20] M. Tyomkyn, An improved bound for the Manickam-Miklós-Singhi conjecture, available online at http://arxiv.org/pdf/1011.2803v1.