Rainbow matchings in properly-colored hypergraphs

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Abstract

A hypergraph H is properly colored if for every vertex $v \in V(H)$, all the edges incident to v have distinct colors. In this paper, we show that if $H_1, \ldots, H_s$ are properly-colored k-uniform hypergraphs on n vertices, where $n \geq 3k^2 s$, and $e(H_i) > \binom{n}{k} - \binom{n-s+1}{k}$, then there exists a rainbow matching of size s, containing one edge from each H_i. This generalizes some previous results on the Erdős Matching Conjecture.

Keywords: rainbow matching, properly-colored hypergraphs

1 Introduction

A *k-uniform hypergraph* is a pair $H = (V, E)$, where $V = V(H)$ is a finite set of vertices, and $E = E(H) \subseteq \binom{V}{k}$ is a family of k-element subsets of V called edges. A *matching* in a hypergraph H is a collection of vertex-disjoint edges. The *size* of a matching is the number of edges in the matching. The matching number $\nu(H)$ is the maximum size of a matching in H. In 1965, Erdős [4] asked to determine the maximum number of edges that could appear in a k-uniform n-vertex hypergraph H with matching number $\nu(H) < s$, for given integer $s \leq \frac{n}{k}$. He conjectured that the problem has two extremal constructions. The first one is a hyper-clique consisting of all the k-subsets on $ks - 1$ vertices. The other one is a k-uniform hypergraph on n vertices containing all the edges intersecting a fixed set of $s - 1$ vertices. Erdős posed the following conjecture:

Conjecture 1.1 ([4]) *Every k-uniform hypergraph H on n vertices with matching number $\nu(H) < s \leq \frac{n}{k}$ satisfies $e(H) \leq \max\{\binom{ks-1}{k}, \binom{n}{k} - \binom{n-s+1}{k}\}$.*

The case $s = 1$ is the classic Erdős–Ko–Rado Theorem [6]. The graph case ($k = 2$) was verified in [5] by Erdős and Gallai. The problem seems to be significantly harder for hypergraphs. When $k = 3$, Frankl, Rödl and Ruciński [11] proved the conjecture for $s \leq \frac{n}{4}$. Łuczak and Mieczkowska [14]

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proved it for sufficiently large s. The $k = 3$ case was finally settled by Frankl [8]. For general k, a short calculation shows that when $s \leq \frac{n}{k+1}$, we always have $\binom{n}{k} - \binom{n-s+1}{k} > \binom{ks-1}{k}$. For this range, the second construction is believed to be optimal. Erdős [4] proved the conjecture for $n \geq n_0(k, s)$. Bollobás, Daykin and Erdős [2] proved the conjecture for $n > 2k^3(s-1)$. Huang, Loh and Sudakov [12] improved it to $n \geq 3k^2 s$, which was further improved to $n \geq 3k^2 s/\log k$ by Frankl, Łuczak and Mieczkowska [10]. On the other hand, in an unpublished note, Füredi and Frankl proved the conjecture for $n \geq cks^2$, Frankl [7] improved all the range above to $n \geq (2s-1)k - s + 1$. Currently the best range is $n \geq \frac{5}{3}sk - \frac{2}{3}s$ by Frankl and Kupavskii [9].

In this paper, we consider a generalization of Erdős Matching Conjecture to properly-colored hypergraphs. A hypergraph H is *properly colored* if for every vertex $v \in V(H)$, all edges incident to v are colored differently. A *rainbow matching* in a properly-colored hypergraph H is a collection of vertex disjoint edges with pairwise different colors. The *size* of a rainbow matching is the number of edges in the matching. The *rainbow matching number*, denoted by $\nu_r(H)$, is the maximum size of a rainbow matching in H. Motivated by the Erdős Matching Conjecture, we consider the following problem: how many edges can appear in a properly-colored k-uniform hypergraph H such that its rainbow matching number satisfies $\nu_r(H) < s \leq \frac{n}{k}$? In fact, it is called Rainbow Turán problem and is well studied in [13]. Note that here if we let H be rainbow, that is, every edge of H receives distinct colors, then we obtain the original Erdős Matching Conjecture.

More generally, let $H_1, \ldots, H_s$ be properly-colored k-uniform hypergraphs on n vertices, a rainbow matching of size s in $H_1, \ldots, H_s$ is a collection of vertex disjoint edges $e_1, \ldots, e_s$ with pairwise different colors, where $e_1 \in E(H_1), \ldots, e_s \in E(H_s)$. For simplicity, we call it an s-rainbow matching. Then what is the minimum M, such that by assuming $e(H_i) > M$ for every i, it guarantees the existance of an s-rainbow matching?

In this paper, we prove the following result, which generalizes Theorem 1.2 and Theorem 3.3 of [12].

Theorem 1.2 *Let $H_1, \ldots, H_s$ be properly-colored k-uniform hypergraphs on n vertices. If $n \geq 3k^2 s$ and every $e(H_i) > \binom{n}{k} - \binom{n-s+1}{k}$, then there exists an s-rainbow matching in $H_1, \ldots, H_s$.*

2 Preliminary results

In this section, we list some preliminary results about "rainbow" hypergraphs, which is a special case of properly-colored hypergraphs. In the next section, we will prove our main theorem with the help of these results. A hypergraph H is *rainbow* if the colors of any two edges in $E(H)$ are different. From now on, when we say an edge e is *disjoint* from a collection of edges, it means that not only e is vertex-disjoint from those edges, but it also has a color different from the colors of all these edges. We start by the following lemma for graphs. Note that here although each G_i is rainbow, a color may appear in more than one G_i's.

Lemma 2.1 *Let $G_1, \ldots, G_s$ be rainbow graphs on n vertices. If $n \geq 5s$ and $e(G_i) > \binom{n}{2} - \binom{n-s+1}{2}$, then there exists an s-rainbow matching in $G_1, \ldots, G_s$.*

Proof. We do induction on s. The base case $s = 1$ is trivial. For every vertex $v \in V(G_i)$ and $j \neq i$, let G_v^j be the subgraph of G_j induced by the vertex set $V(G_j) \setminus \{v\}$. Since there are at most $n-1$ edges containing v in $E(G_j)$, we have $e(G_v^j) \geq e(G_j) - (n-1) > \binom{n}{2} - \binom{n-s+1}{2} - (n-1) = \binom{n-1}{2} - \binom{(n-1)-(s-1)+1}{2}$. By induction, there exists an $(s-1)$-rainbow matching $\{e_j\}_{j\neq i}$ in $\{G_v^j\}_{j\neq i}$, which spans $2(s-1)$ vertices. So if some G_i has a vertex v with degree greater than $3(s-1)$, then there exists an edge e in G_i which contains v and disjoint from the edges of the $(s-1)$-rainbow matching, which produces an s-rainbow matching. Hence we may assume that the maximum degree of each G_i is at most $3(s-1)$.

Now pick an arbitrary edge uv in G_1. Assume the color of uv is $c(uv)$. Then we delete the vertices u, v and the edge colored by $c(uv)$ in $G_2, \ldots, G_s$. Denote the resulting graphs by $G_2', \ldots, G_s'$. We can see that when $n \geq 5s$, for each $i \in \{2, \ldots, s\}$, we have $e(G_i') > \binom{n}{2} - \binom{n-s+1}{2} - 2\cdot 3(s-1) - 1 > \binom{n-2}{2} - \binom{(n-2)-(s-1)+1}{2}$. By induction on s, there exists an $(s-1)$-rainbow matching in the graphs $G_2', \ldots, G_s'$. Taking these $s-1$ edges with the edge uv, we obtain an s-rainbow matching in G_1, $\ldots$, G_s.

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Lemma 2.2 *Let $H_1, \ldots, H_s$ be rainbow k-uniform hypergraphs on n vertices. If $n \geq 3k^2 s$ and $e(H_i) > \binom{n}{k} - \binom{n-s+1}{k}$, then there exists an s-rainbow matching in $H_1, \ldots, H_s$.*

Proof. We do induction on both k and s. According to Lemma 2.1, the case $k = 2$ holds for every s and $n \geq 5s$. And for every k, the case $s = 1$ is trivial. We first consider the situation when some H_i has a vertex v with degree greater than $k(s-1)\binom{n-2}{k-2} + s - 1$. For every vertex $v \in V(H_i)$ and $j \neq i$, let H_v^j be the subgraph of H_j induced by the vertex set $V(H_j) \setminus \{v\}$. Since there are at most $\binom{n-1}{k-1}$ edges containing v in $E(H_j)$, we have $e(H_v^j) \geq e(H_j) - \binom{n-1}{k-1} > \binom{n}{k} - \binom{n-s+1}{k} - \binom{n-1}{k-1} = \binom{n-1}{k} - \binom{(n-1)-(s-1)+1}{k}$. By inductive hypothesis for the case $(n-1, k, s-1)$, there exists an $(s-1)$-rainbow matching $\{e_j\}_{j\neq i}$ in $\{H_v^j\}_{j\neq i}$, which spans $k(s-1)$ vertices. So if some H_i has a vertex v with degree greater than $k(s-1)\binom{n-2}{k-2} + s - 1$, then there exists an edge e in $E(H_i)$ which contains v and disjoint from the edges of the $(s-1)$-rainbow matching, which produces an s-rainbow matching. Hence we may assume that the maximum degree in each hypergraph H_i is at most $k(s-1)\binom{n-2}{k-2} + s - 1$.

By induction on s, we know that for every i there exists an $(s-1)$-rainbow matching in the hypergraphs $\{H_j\}_{j\neq i}$, spanning $k(s-1)$ vertices. If for some i, the s-th largest degree of H_i is at most $2(s-1)\binom{n-2}{k-2}+s-1$, then the sum of degrees of these $k(s-1)$ vertices in H_i is at most

$$(s-1)[k(s-1)\tbinom{n-2}{k-2}+s-1]+(s-1)(k-1)[2(s-1)\tbinom{n-2}{k-2}+s-1] = (3k-2)(s-1)^2\tbinom{n-2}{k-2}+(s-1)^2k.$$

Since $n \geq 3k^2s$, we have $e(H_i) > \binom{n}{k}-\binom{n-s+1}{k} > (s-1)^2(3k-\frac{1}{2})\binom{n-2}{k-2} > (3k-2)(s-1)^2\binom{n-2}{k-2}+(s-1)^2k+s-1$, which guarantees the existence of an edge in H_i which is disjoint from the previous $(s-1)$-rainbow matching in $\{H_j\}_{j\neq i}$, which produces an s-rainbow matching. So we may assume that each H_i contains at least s vertices with degree above $2(s-1)\binom{n-2}{k-2}+s-1$.

Now we may greedily select distinct vertices $v_i \in V(H_i)$, such that for each $1 \leq i \leq s$, the degree of v_i in H_i exceeds $2(s-1)\binom{n-2}{k-2}+s-1$. Consider all the subsets of $V(H_i)\setminus\{v_1,\ldots,v_s\}$ which together with v_i form an edge of H_i. Denote the $(k-1)$-uniform hypergraph by H_i'. Then $e(H_i') > 2(s-1)\binom{n-2}{k-2}+s-1-(s-1)\binom{n-2}{k-2} > \binom{n-s}{k-1}-\binom{n-2s+1}{k-1}$. By the inductive hypothesis for the case $(n-s,k-1,s)$, there exists an s-rainbow matching $\{e_i\}_{1\leq i\leq s}$ in $\{H_i'\}_{1\leq i\leq s}$. Taking the edges $e_i\bigcup\{v_i\}$, we obtain an s-rainbow matching in $\{H_i\}_{1\leq i\leq s}$.

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3 Main Theorem

In this section we prove our main result, Theorem 1.2, using induction and Lemma 2.2.

Proof. We split our proof into two cases.

Case 1: $k = 2$**.** Now $H_1, \ldots, H_s$ are properly-colored graphs. We do induction on s. The base case $s=1$ is trivial. For every vertex $v \in V(H_i)$ and $j \neq i$, let H_v^j be the subgraph of H_j induced by the vertex set $V(H_j)\setminus\{v\}$. Since there are at most $n-1$ edges containing v in $E(H_j)$, we have $e(H_v^j) \geq e(H_j)-(n-1) > \binom{n}{2}-\binom{n-s+1}{2}-(n-1) = \binom{n-1}{2}-\binom{(n-1)-(s-1)+1}{2}$. By induction, there exists an $(s-1)$-rainbow matching $\{e_j\}_{j\neq i}$ in $\{H_v^j\}_{j\neq i}$, which spans $2(s-1)$ vertices. So if some H_i has a vertex v of degree greater than $3(s-1)$, then there exists an edge e in H_i which contains v and disjoint from the edges of the $(s-1)$-rainbow matching, which produces an s-rainbow matching. Hence we may assume the maximum degree in each H_i is at most $3(s-1)$.

For every color c in H_i and $j \neq i$, let H_c^j be the subgraph of H_j obtained by deleting all the edges colored by c in $E(H_j)$. Since each H_j is properly colored, there are at most $\frac{n}{2}$ edges colored by c in $E(H_j)$. So $e(H_c^j) \geq e(H_j)-\frac{n}{2} > \binom{n}{2}-\binom{n-s+1}{2}-\frac{n}{2} > \binom{n}{2}-\binom{n-(s-1)+1}{2}$. By induction, there exists an $(s-1)$-rainbow matching $\{e_j\}_{j\neq i}$ in $\{H_v^j\}_{j\neq i}$, which spans $2(s-1)$ vertices $u_1,\ldots,u_{2(s-1)}$. Also since H_i is properly colored, it has at most one edge containing each u_j and colored by c. So if the number of edges in H_i colored by c is greater than $2(s-1)$, then there exists an edge e in

H_i colored by c and disjoint from $\{e_j\}_{j\neq i}$, which produces an s-rainbow matching. So we can now assume that the number of edges in every color in each H_i is at most $2(s-1)$.

Now pick an arbitrary edge uv in H_1. Assume the color of uv is $c(uv)$. Then we delete the vertices u, v and all the edges colored by $c(uv)$ in $H_2, \dots, H_s$. Denote the resulting graphs by $H'_2, \dots, H'_s$. We can see that when $n \geq 7s$, for each $i \in \{2, \dots, s\}$, we have $e(H'_i) > \binom{n}{2} - \binom{n-s+1}{2} - 2\cdot 3(s-1) - 2(s-1) > \binom{n-2}{2} - \binom{(n-2)-(s-1)+1}{2}$. By induction on s, there exists an $(s-1)$-rainbow matching in the graphs $H'_2, \dots, H'_s$. Taking these $s-1$ edges with the edge uv, we obtain an s-rainbow matching in $H_1, \dots, H_s$.

Case 2: $k \geq 3$. We do induction on s. The case $s=1$ is trivial. We first consider the situation when some H_i has a vertex of degree greater than $k(s-1)\binom{n-2}{k-2} + s - 1$. For every vertex $v \in H_i$ and $j \neq i$, let H_v^j be the subgraph of H_j induced by the vertex set $V(H_j) \setminus \{v\}$. Since there are at most $\binom{n-1}{k-1}$ edges containing v in $E(H_j)$, we have $e(H_v^j) \geq e(H_j) - \binom{n-1}{k-1} > \binom{n}{k} - \binom{n-s+1}{k} - \binom{n-1}{k-1} = \binom{n-1}{k} - \binom{(n-1)-(s-1)+1}{k}$. By induction, there exists an $(s-1)$-rainbow matching $\{e_j\}_{j\neq i}$ in $\{H_v^j\}_{j\neq i}$, which spans $k(s-1)$ vertices. So if some H_i has a vertex v with degree greater than $k(s-1)\binom{n-2}{k-2} + s - 1$, then there exists an edge e in $E(H_i)$ which contains v and disjoint from the edges of the $(s-1)$-rainbow matching, which produces an s-rainbow matching. Hence we may assume the maximum degree in each hypergraph H_i is at most $k(s-1)\binom{n-2}{k-2} + s - 1$.

By induction on s, we know that for every i there exists an $(s-1)$-rainbow matching in the hypergraphs $\{H_j\}_{j\neq i}$, spanning $k(s-1)$ vertices. If for some i, the s-th largest degree of H_i is at most $2(s-1)\binom{n-2}{k-2} + s - 1$, then the sum of degrees of these $k(s-1)$ vertices in H_i is at most

$$(s-1)[k(s-1)\tbinom{n-2}{k-2}+s-1]+(s-1)(k-1)[2(s-1)\tbinom{n-2}{k-2}+s-1] = (3k-2)(s-1)^2\tbinom{n-2}{k-2}+(s-1)^2k.$$

On the other hand, the maximum degree of the subgraph of H_i by deleting these $k(s-1)$ vertices is at most $s-1$, otherwise, we can find an s-rainbow matching. Since $n \geq 3k^2 s$, we have $e(H_i) > \binom{n}{k} - \binom{n-s+1}{k} > (s-1)^2(3k - \frac{1}{2})\binom{n-2}{k-2} > (3k-2)(s-1)^2\binom{n-2}{k-2} + (s-1)^2 k + \frac{(s-1)[n-k(s-1)]}{k}$, which guarantees the existence of an edge in H_i disjoint from the previous $(s-1)$-rainbow matching in $\{H_j\}_{j\neq i}$, which produces an s-rainbow matching. So we may assume that each H_i contains at least s vertices with degree above $2(s-1)\binom{n-2}{k-2} + s - 1$.

Now we may greedily select distinct vertices $v_i \in V(H_i)$, such that for each $1 \leq i \leq s$, the degree of v_i in H_i exceeds $2(s-1)\binom{n-2}{k-2}+s-1$. Consider all the subsets of $V(H_i)\setminus\{v_1, \dots, v_s\}$ which together with v_i form an edge of H_i. Denote the $(k-1)$-uniform hypergraph by H'_i. Since each H_i is properly colored, we can see that each H'_i is rainbow and $e(H'_i) > 2(s-1)\binom{n-2}{k-2} + s - 1 - (s-1)\binom{n-2}{k-2} > \binom{n-s}{k-1} - \binom{n-2s+1}{k-1}$. By Lemma 2.2, there exists an s-rainbow matching $\{e_i\}_{1\leq i\leq s}$ in $\{H'_i\}_{1\leq i\leq s}$. Taking the edges $e_i \bigcup \{v_i\}$, we obtain an s-rainbow matching in $\{H_i\}_{1\leq i\leq s}$.

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4 Concluding Remarks

In this short note, we propose a generalization of the Erdős hypergraph matching conjecture to finding rainbow matchings in properly-colored hypergraphs, and prove Theorem 1.2 for $s < n/(3k^2)$. The following conjecture seems plausible.

Conjecture 4.1 *There exists constant $C > 0$ such that if $H_1, \ldots, H_s$ are properly-colored k-uniform hypergraphs on n vertices, with $n \geq Cks$ and every $e(H_i) > \binom{n}{k} - \binom{n-s+1}{k}$, then there exists an s-rainbow matching in $H_1, \ldots, H_s$.*

Recall that for the special case when each H_i is identical and rainbow, Frankl and Kupavskii [9] were able to verify it for $C = 5/3$. However the proof relies on the technique of shifting, while the property of a hypergraph being properly colored may not be preserved under shifting.

It is tempting to believe that Erdős Matching Conjecture can be extended to properly-colored hypergraphs for the entire range of s, that is, once the number of edges in each hypergraph exceeds the maximum of $\binom{n}{k} - \binom{n-s+1}{k}$ and $\binom{ks-1}{k}$, then one can find an s-rainbow matching. However this is false in general, a simple construction is by taking $s = 2$ and $n = 2k$. The maximum of these two expressions is $\binom{2k-1}{k}$, while one can let H_1 be a rainbow K_{2k}^k with an edge coloring c_1, and H_2 be on the same vertex set with edge coloring c_2, such that $c_2(e) = c_1([2k] \setminus e)$. Then clearly each H_i contains $\binom{2k}{k} > \binom{2k-1}{k}$ edges and there is no 2-rainbow matching. It would be interesting to find constructions for s close to n/k, and formulate a complete conjecture for properly-colored hypergraphs.

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