On local Turán problems

Peter Frankl * Hao Huang † Vojtěch Rödl ‡

Abstract

Since its formulation, Turán’s hypergraph problems have been among the most challenging open problems in extremal combinatorics. One of them is the following: given a 3-uniform hypergraph $F$ on $n$ vertices in which any five vertices span at least one edge, prove that $|F| \geq (1/4-o(1))\binom{n}{3}$. The construction showing that this bound would be best possible is simply $\left(\binom{X}{3}\right) \cup \left(\binom{Y}{3}\right)$ where $X$ and $Y$ evenly partition the vertex set. This construction has the following more general $(2p+1, p+1)$-property: any set of $2p+1$ vertices spans a complete sub-hypergraph on $p+1$ vertices. One of our main results says that, quite surprisingly, for all $p > 2$ the $(2p+1, p+1)$-property implies the conjectured lower bound.

1 Introduction

Let $X$ be a finite set and $\binom{X}{r}$ the collection of all its $r$-subsets. Subsets $H$ of $\binom{X}{r}$ are called $r$-uniform hypergraphs. Members of $H$ are called edges. If $\binom{Y}{r} \subset H$, then $Y$ is said to be a clique and $|Y|$ is its size. We denote by $K^r_t$ the $r$-uniform $t$-vertex clique. Note that every edge is a clique of size $r$.

For integers $q \geq p \geq r \geq 2$, we say that $H$ has property $(q,p)$ if for every $Z \in \binom{X}{q}$ there exists $Y \subset \binom{Z}{p}$ spanning a clique in $H$, that is, $\binom{Y}{r} \subset H$.

Definition 1.1. Let $T_r(n, q, p) = \min\{|H| : H \subset \binom{[n]}{r}, H \text{ has property } (q, p)\}$. Set also $t_r(n, q, p) = T_r(n, q, p) / \binom{n}{r}$.

Eighty years ago, Turán [9] determined $T_2(n, q, 2)$ and this result served as the starting point for a lot of research that led to the creation of the field of extremal graph theory. About two decades later Turán [10] proposed two conjectures concerning $T_3(n, 4, 3)$ and $T_3(n, 5, 3)$. To state their asymptotic forms, let us mention that Katona,
Nemetz and Simonovits [6] used a simple averaging argument to show that $t_r(n,q,p)$ is monotone increasing as a function of $n$. Consequently the limit

$$\lim_{n \to \infty} t_r(n,q,p) =: t_r(q,p)$$

exists.

**Conjecture 1.2.** (Turán)

$$t_3(4,3) = \frac{4}{9}. \quad (1)$$

$$t_3(5,3) = \frac{1}{4}. \quad (2)$$

Even though this conjecture has been around for quite a long time, neither statement was proved. For (1) the best known bound stands as $t_3(4,3) \geq 0.438334$ by Razborov [8] using flag algebra. As for (2), the construction providing the upper bound is very simple, namely $H = \binom{X_1}{r} \cup \binom{X_2}{r}$, with $X_1 \sqcup X_2 = [n]$, $|X_1| = \lceil \frac{n}{2} \rceil$, $|X_2| = \lfloor \frac{n}{2} \rfloor$.

Let us mention that in [2] it was shown that for the graph case,

$$t_2(q,p) = 1/\left\lfloor \frac{q-1}{p-1} \right\rfloor. \quad (3)$$

For general $r$, Frankl and Stechkin [4] proved that

$$t_r(q,p) = 1 \quad \text{if} \quad q \leq \frac{r}{r-1}(p-1). \quad (4)$$

It is easy to check that $H = \binom{X_1}{r} \cup \binom{X_2}{r}$ has property $(2p+1,p+1)$ for all $p \geq r-1$. Consequently,

$$t_r(2p+1,p+1) \leq \frac{1}{2^{r-1}}. \quad (5)$$

For the case $r = 3$, it was proved by the first author [3] that

$$\lim_{p \to \infty} t_3(2p+1,p+1) = \frac{1}{4}. \quad (6)$$

By developing the methods used in [3], in Section 2 we generalize (6) to the $r$-uniform case.

**Theorem 1.3.** For integers $r \geq 2$ and $a \geq 2$,

$$\lim_{p \to \infty} t_r(ap+1,p+1) = \frac{1}{a^{r-1}}.$$

In the 3-uniform case (when $r = 3$), we are able to determine the exact value of $t_3(2p+1,p+1)$, for all $p \geq 3$, which strengthens (6).

**Theorem 1.4.** For every integer $p \geq 3$,

$$t_3(2p+1,p+1) = \frac{1}{4}.$$
We should remark that the proof of this result is relying on earlier Turán-type results of Mubayi and Rödl [7], and Baber and Talbot [1]. We are going to state these results in Section 3 before proving Theorem 1.4. In Section 4 we mention some open problems.

2 Proof of Theorem 1.3

Throughout the proof of Theorem 1.3, we assume \( r \geq 3 \), and \( a \geq 2 \) to be fixed, since the \( r = 2 \) case is already covered by (3). With \( r \) fixed, we also set \( t(q,p) = t_r(q,p) \). For the pair \((q,p)\) with \( q \leq ap \), we call \( ap - q \) the excess \( e(q,p) \) of the pair \((q,p)\). Note that since \( q \geq p \), we always have \( e(q,p) \leq ap - q = (a - 1)q \). For \( F \subset \binom{Y}{q} \), the set \( Z \) is a \((w,v)\)-hole if \(|Z| = w\), the clique number of \( F|_Z \) (the sub-hypergraph of \( F \) induced by \( Z \)) is \( v \), and \( w > av \). We first establish the following two lemmas.

**Lemma 2.1.** Suppose \( G \subset \binom{Y}{r} \) has property \((q,p)\), and \( Z \) is a \((w,v)\)-hole of \( G \) with \( w < q \), then \( G|_{Y \setminus Z} \) has property \((q - w,p - v)\).

**Proof.** Take an arbitrary set \( U \in \binom{Y \setminus Z}{q - w} \), then \( U \cup Z \in \binom{Y}{q} \). Since \( G \) has property \((q,p)\), \( G|_{U \cup Z} \) contains a clique of size \( p \). Hence \( G|_U \) contains a clique of size \( p - v \). \( \square \)

**Lemma 2.2.** Suppose an \( r \)-uniform hypergraph \( F \) has property \((q,p)\) for all pairs \((q,p)\) with \( q \leq a\ell \) and \( p = \lceil q/a \rceil \) (in other words \( F \) does not have a \((w,v)\)-hole with \( a\ell \geq w > av \)). Then for all \( Y \subset \binom{X}{a\ell} \),

\[
\left| F \cap \binom{Y}{r} \right| \geq a \binom{\ell}{r}.
\]

**Proof.** Instead of this we prove the following stronger statement. Let \((r - 1)a \leq s \leq a\ell \) and \( Y \in \binom{X}{s} \). Suppose further that \( s = (a - b)t + b(t - 1) \) for some \( 0 \leq b < a \), then

\[
\left| F \cap \binom{Y}{r} \right| \geq (a - b) \binom{t}{r} + b \binom{t - 1}{r}.
\]

Note that the right hand side is 0 when \( s \leq (r - 1)a \), so the inequality is trivially true in this range. To prove the general case, we use induction on \( s \). Since \( s = (a - b)t + b(t - 1) \in \{at - a + 1, \ldots, at\} \), \( F \) has the \((s,t)\) property from the assumption. Let \( R \in \binom{Y}{t} \) span a clique and fix \( y \in R \). There are \( \binom{t - 1}{r - 1} \) edges in \( \binom{R}{r} \cap F \) containing \( y \). Remove \( y \) from \( F \) and apply the inductive hypothesis to \( F \setminus \{y\} \). We infer that

\[
\left| F \cap \binom{Y \setminus \{y\}}{r} \right| \geq (a - b - 1) \binom{t}{r} + (b + 1) \binom{t - 1}{r}.
\]

Considering the at least \( \binom{t - 1}{r - 1} \) edges containing \( y \), we have

\[
\left| F \cap \binom{Y}{r} \right| \geq (a - b - 1) \binom{t}{r} + (b + 1) \binom{t - 1}{r} + \binom{t - 1}{r - 1}.
\]

\[
= (a - b) \binom{t}{r} + b \binom{t - 1}{r}.
\]

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Now we can proceed as follows to prove Theorem 1.3. The upper bound \( \lim_{p \to \infty} t_r(ap+1, p+1) \leq \frac{1}{a^r-1} \) is immediate, since \( \mathcal{H}_{n,r,a} := \binom{X}{r} \cup \cdots \cup \binom{X_n}{r} \) with \( X = [n] \) and \( X_i \in \{ [n/a], [n/a] \} \) has property \( (ap+1,p+1) \) and edge density \( 1/a^r-1 + o(1) \). For the remaining of this section we focus on proving the lower bound.

Given \( \varepsilon > 0 \), let us fix a large integer \( \ell > \ell_0(a, r, \varepsilon) \), to be determined later. Then fix a much larger integer \( L \geq 2a^3\ell^2 \), and consider a sufficiently large \( r \)-uniform hypergraph \( \mathcal{F}_0 \subset \binom{[n]}{r} \) having property \((q,p)\) with \( q = aL, \ p = L \). Our aim is to find a subset \( X \subset [n] \) with \( |X_r| > (1 - \varepsilon/2)\binom{n}{r} \) such that \( \mathcal{F}_0 \cap \binom{X}{r} \) has no \( (w,v) \)-hole with \( w \leq a\ell \) and \( r - 1 \leq v \).

To this end, we start with \( \mathcal{F}_0 \) and define \( \mathcal{F}_i \) inductively. Let \( q_0 = q, p_0 = p, X_0 = [n] \). Suppose that \( \mathcal{F}_i \subset \binom{X_i}{r} \) has property \((q_i,p_i)\) and it still has a \((w_i,v_i)\)-hole. Then we let \( Z_i \subset X_i \) be such a \((w_i,v_i)\)-hole, and set

\[
X_{i+1} = X_i \setminus Z_i, \quad \mathcal{F}_{i+1} = \mathcal{F}_i \cap \binom{X_{i+1}}{r}.
\]

By Lemma 2.1, \( \mathcal{F}_{i+1} \) has property \((q_i-w_i, p_i-v_i)\). Moreover, the new excess satisfies

\[
e(q_i - w_i, p_i - v_i) = a(p_i - v_i) - (q_i - w_i) = (ap_i - q_i) - (av_i - w_i) \geq e(q_i, p_i) + 1.
\]

Set \( q_{i+1} = q_i - w_i, \ p_{i+1} = p_i - v_i \) and continue. At every step \( a(r - 1) \leq av_i < |X_i| - |X_{i-1}| = w_i \leq a\ell \).

Suppose at step \( i \), the hypergraph \( \mathcal{F}_i \) no longer contains a \((w,v)\)-hole with \( w \leq a\ell \). In this case, we choose a subset \( Q \) of size \( a\ell \) of \( V(\mathcal{F}_i) \) uniformly at random. Then by Lemma 2.2,

\[
\frac{|\mathcal{F}_i|}{\binom{X_i}{r}} = \frac{E|\mathcal{F}_i \cap \binom{Q}{r}|}{\binom{a\ell}{r}} \geq \frac{a\ell}{\binom{r}{r}}.
\]

For sufficiently large \( \ell > \ell_0(a, r, \varepsilon) \), this quantity is greater than \( (1 - \varepsilon/2) \cdot \frac{1}{a^r-1} \). On the other hand, \( |X_i| \geq n - ial \geq n - pal/(r-1) \). Therefore when \( n \) is sufficiently large, \( |\binom{X_i}{r}| > (1 - \varepsilon/2)\binom{n}{r} \) and therefore

\[
|\mathcal{F}_0| \geq |\mathcal{F}_i| \geq (1 - \varepsilon/2) \cdot \frac{1}{a^r-1} \left( \frac{|X_i|}{r} \right) \geq (1 - \varepsilon) \cdot \frac{1}{a^r-1} \binom{n}{r}.
\]

Otherwise suppose this process continues to produce \((w,v)\)-holes. let \( m \) be the first index such that \( q_m < 2a\ell \). In view of \( e(q_m, p_m) \leq (a-1)q_m \) and that \( e(q_i, p_i) \) strictly increases after each step, \( m \leq (a-1)q_m \) follows. Thus

\[
aL = q_0 = q_m + \sum_{i=0}^{m-1} w_i \leq 2a\ell + mal \leq 2a\ell + (a-1) \cdot 2a\ell \cdot a\ell < 2a^3\ell^2,
\]

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contradicting \( L \geq 2a^3\ell^2 \).

Summarizing the two cases above, we have that \( \lim_{L \to \infty} t_r(aL, L) \geq 1/a^{r-1} \). Note that a hypergraph having property \((aL + 1, L + 1)\) must also have property \((aL, L)\). Therefore,
\[
\lim_{p \to \infty} t_r(ap + 1, p + 1) \geq 1/a^{r-1}.
\]
Together with the construction in the introduction that gives \( t_r(ap + 1, p + 1) \leq 1/a^{r-1} \), we conclude the proof of Theorem 1.3.

**Remark.** Since \( \mathcal{H}_{n,r,a} \) also has property \((ap, p)\), we have actually proved a result slightly stronger than Theorem 1.3, namely for every \( a, r \geq 2 \),
\[
\lim_{p \to \infty} t_r(ap, p) = \frac{1}{a^{r-1}}.
\]

### 3 The 3-uniform case

Note that Theorem 1.3, when applied to \( a = 2 \), gives
\[
\lim_{p \to \infty} t_r(2p + 1, p + 1) = \frac{1}{2^{r-1}}.
\]
In this section, we determine the exact value of \( t_r(2p + 1, p + 1) \) for \( r = 3 \) and all \( p \geq 3 \), establishing Theorem 1.4. Our proof is based on two previously known Turán-type results. To apply them, let us change to the complementary notion of excluded configuration.

**Definition 3.1.** For an \( r \)-uniform hypergraph \( F \subset \binom{[n]}{r} \). Let \( \alpha(F) \) be its independence number, that is, \( \alpha(F) = \max\{|A| : A \subset [n], F \cap \binom{A}{r} = \emptyset\} \).

Let \( F^c = \binom{[n]}{r} \setminus F \) be the complementary \( r \)-uniform hypergraph. Now \( F \) has property \((q, p)\) if and only if \( \alpha(H) \geq p \) for all induced sub-hypergraphs \( H = F^c \cap \binom{Q}{p} \), \( Q \subset [n], |Q| = q \).

For a collection of \( G_1, \ldots, G_s \) of \( r \)-uniform hypergraphs, let
\[
t(n, G_1, \ldots, G_s) = \max\{|F| : F \subset \binom{[n]}{r}, F \text{ contains no copy of } G_i, i = 1, \ldots, s\}.
\]
It is easily seen that \( t(n, G_1, \ldots, G_s)/\binom{n}{r} \) is a monotone decreasing function of \( n \). Consequently \( \lim_{n \to \infty} t(n, G_1, \ldots, G_s)/\binom{n}{r} \) exists. This limit is denoted by \( \pi(G_1, \ldots, G_s) \), and it is usually called the Turán density of \( \{G_1, \ldots, G_s\} \).

Consider the following three hypergraphs from [7]:
\[
\mathcal{R}_0 = \binom{[4]}{3} \cup \{(a, x, y) : a \in [4], x, y \in \{5, 6, 7\}, x \neq y\},
\]
\[
\mathcal{R}_1 = \mathcal{R}_0 \setminus \{\{1, 5, 6\}, \{2, 5, 7\}, \{3, 6, 7\}\},
\]
\[ R_2 = R_0 \setminus \{\{1, 5, 6\}, \{1, 5, 7\}, \{3, 6, 7\}\}. \]

It is easy to check that \( \alpha(R_i) = 3 \) for \( i = 0, 1, 2 \). To prove \( t_3(7, 4) = 1/4 \), it suffices to prove
\[
\pi(R_1, R_2) = \frac{3}{4}. \tag{7}
\]

Actually Mubayi and the third author [7] proved a considerably stronger statement. Set \( R = R_0 \setminus \{1, 5, 6\} \). Then

**Proposition 3.2.** (\([7]\)) \( \pi(R) = \frac{3}{4} \).

Since the proof of Proposition 3.2 is rather short let us include it. Suppose that \( \varepsilon > 0, n > n_0(\varepsilon) \) and \( H \subset \binom{[n]}{3} \) satisfies \( |H| \geq (3/4 + \varepsilon)(\binom{n}{3}) \). Then for a 4-element set \( Y \subset [n] \) chosen uniformly at random, the expected size of \( |H \cap \binom{Y}{3}| = 4|H|/(\binom{n}{3}) \geq 3 + \varepsilon \).

Consequently, \( H \) contains many complete 3-uniform hypergraphs on 4 vertices. (As a matter of fact, instead of 3/4 to ensure that, Razborov [8] proved that 0.516⋯ would be sufficient to ensure the existence of \( K_4^3 \).) By symmetry, suppose \( \binom{[4]}{4} \subset H \).

For \( i \in [4] \) define the link graphs \( H(i) = \{(x, y) \subset [5, n] : (i, x, y) \in H\} \). Let \( G \) be the multigraph whose edge set is the union (with multiplicities) \( H(1) \cup \cdots \cup H(4) \). Should \( |G| > 3\binom{n-4}{2} + n - 6 \) hold, we can apply a result of Füredi and Kündgen [5] which guarantees that there are three vertices in \( G \) spanning at least 11 edges, which corresponds to a copy of \( R \) in \( H \). In the opposite case \( |H(i)| < (3/4 + \varepsilon/2)(\binom{n}{3}) \) for some \( i \in [4] \), then we remove the vertex \( i \) and iterate. Either we find \( R \) or we arrive at a contradiction with \( |H| > (3/4 + \varepsilon)(\binom{n}{3}) \).

The following result was proved by Barber and Talbot [1] using flag algebra.

**Proposition 3.3.** (Theorem 18 in [1]) Let \( T \) be the 6-vertex 3-uniform vertex hypergraph with
\[
T = \binom{\{6\}}{3} \setminus \{\{1, 5, 6\}, \{2, 4, 6\}, \{2, 5, 6\}, \{3, 4, 6\}, \{3, 4, 5\}\}.
\]

Then \( \pi(T) = 3/4 \).

Now we are ready to prove Theorem 1.4. Observe that if \( G \) and \( H \) are two hypergraphs and \( F \) is their vertex-disjoint union, then \( \pi(F) = \max\{\pi(G), \pi(H)\} \).

**Proof of Theorem 1.4.** We have the upper bound \( t_3(2p + 1, p + 1) \leq 1/4 \) from (5). Therefore it suffices to establish a matching lower bound. By considering the complement of the host hypergraph, it boils down to showing that if the edge density of a 3-uniform hypergraph \( G \) is greater than \( 3/4 + o(1) \), then \( G \) contains a sub-hypergraph \( H \) on \( 2p + 1 \) vertices with \( \alpha(H) \leq p \). In other words, we need \( \pi(H) \leq 3/4 \).

For odd \( p \geq 3 \), we let \( H_1 \) be the vertex-disjoint union of \( R \) and \( (p - 3)/2 \) copies of \( K_4^3 \). It is straightforward to check that \( H_1 \) has \( 7 + 4 \cdot (p - 3)/2 = 2p + 1 \) vertices, independence number \( 3 + (p - 3) = p \), and \( \pi(H_1) = \max\{\pi(R), \pi(K_4^3)\} = 3/4 \). This gives \( t_3(2p + 1, p + 1) \geq 1/4 \) for all odd \( p \geq 3 \).
For even $p \geq 4$, we take $\mathcal{T}$ from Lemma 3.3, and blow up its vertices $1, 2, 3$ twice, and vertices $4, 5, 6$ once to obtain a 9-vertex hypergraph $\mathcal{T'}$. Note that a blow-up could only have lower Turán density, therefore $\pi(\mathcal{T'}) \leq \pi(\mathcal{T}) = 3/4$. Moreover the independence number of $\mathcal{T'}$ is $4$, since all the five non-edges of $\mathcal{T}$ contain at most one vertex from $\{1, 2, 3\}$ and $\{4, 5, 6\}$ itself is an edge. We then let $\mathcal{H}_2$ be the vertex-disjoint union of $\mathcal{T'}$ with $(p - 4)/2$ copies of $K_4^3$. Then $\mathcal{H}_2$ has $9 + 4 \cdot (p - 4)/2 = 2p + 1$ vertices, $\alpha(\mathcal{H}_2) = 4 + (p - 4) = p$, and $\pi(\mathcal{H}_2) = \max\{\pi(\mathcal{T'}), \pi(K_4^3)\} \leq 3/4$. Therefore for all even $p \geq 4$, we also have $t(2p + 1, p + 1) \geq 1/4$. This completes the proof. \hfill \square

4 Concluding Remarks

It is tempting to conjecture that the Turán density $\pi(K_{2r-1}^r) = 1 - 1/2^{r-1}$. This is Mantel’s Theorem for $r = 2$. For $r = 3$, it is a special case (when $k = 5$) of Turán’s well-known conjecture: for every $k \geq 4$, $\pi(K_k^3) = 1 - 4/(k - 1)^2$. If this is true, then we would also have $\pi(K_{2r-2}^r) \leq 1 - 1/2^{r-1}$, therefore if an $r$-uniform hypergraph has edge density at least $1 - 1/2^{r-1} + o(1)$, it must contain a copy of $\mathcal{H}$ which is the vertex-disjoint union of $K_{2r-1}^r$ and $t$ copies of $K_{2r-2}^r$. It is easy to verify that $\mathcal{H}$ has $2r - 1 + t(2r - 2)$ vertices and independence number $(t + 1)(r - 1)$. Therefore $\pi(K_{2r-1}^r) = 1/2^{r-1}$ would imply $t(2p + 1, p + 1) = 1/2^{r-1}$ for $p = (t + 1)(r - 1)$ for all integers $t \geq 1$. Maybe even the following stronger conjecture is true. Our Theorem 1.3 indicates this is true in the limit, and Theorem 1.4 settles the $r = 3$ case except for $p = 2$ (which corresponds to $K_5^3$).

**Conjecture 4.1.** For integers $r \geq 2$, $p \geq r - 1$,

$$t_r(2p + 1, p + 1) = \frac{1}{2^{r-1}}.$$

Here we remark that $\mathcal{T}$ in Lemma 3.3 with the edge $\{1, 4, 5\}$ removed still has all the properties needed for the proof of Theorem 1.4. Perhaps one could find a simpler proof that this new hypergraph, much more symmetric than $\mathcal{T}$, still has Turán density $3/4$. Such proof might provide some new insights on the above conjecture.

To determine $t_r(q, p)$, we essentially seek $r$-uniform hypergraph $\mathcal{H}$ with low independence number $\alpha(\mathcal{H})$ relative to its number of vertices, and low Turán density $\pi(\mathcal{H})$. In light of this observation and the results (3) and (4), could it possibly be true that for every positive real number $\gamma > 0$,

$$\lim_{p \to \infty} t_r(\gamma p + 1, p + 1) = 1 - \min_{\mathcal{H} \in \mathcal{F}} \pi(\mathcal{H}) = 1/|\gamma|^r,$$

where $\mathcal{F}$ is family of all the $r$-uniform hypergraph satisfying $|V(\mathcal{H})| \geq \gamma \alpha(\mathcal{H})$?

Finally, motivated by the asymptotic result (7) we propose the following conjecture:

**Conjecture 4.2.** There exists $n_0$ such that for all integers $n > n_0$,

$$t(2n, \mathcal{R}_1, \mathcal{R}_2) = \binom{2n}{3} - 2\binom{n}{3}.$$
References


