REAL COHOMOLOGY AND THE POWERS OF THE FUNDAMENTAL IDEAL IN THE WITT RING

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Abstract. Let $A$ be a local ring with $2$ invertible. It is known that the localization of the cohomology ring $H^\ast_{\acute{e}t}(A, \mathbb{Z}/2)$ with respect to the class $(-1) \in H^1_{\acute{e}t}(A, \mathbb{Z}/2)$ is isomorphic to the ring $C(\operatorname{sper} A, \mathbb{Z}/2)$ of continuous $\mathbb{Z}/2$-valued functions on the real spectrum of $A$. Let $I^n(A)$ denote the powers of the fundamental ideal in the Witt ring of symmetric bilinear forms over $A$.

The starting point of this article is the “integral” version: the localization of the graded ring $\bigoplus_{n \geq 0} I^n(A)$ with respect to the class $\langle\langle -1 \rangle\rangle := \langle 1, 1 \rangle \in I(A)$ is isomorphic to the ring $C(\operatorname{sper} A, \mathbb{Z})$ of continuous $\mathbb{Z}$-valued functions on the real spectrum of $A$.

This has interesting applications to schemes. For instance, for any algebraic variety $X$ over the field of real numbers $\mathbb{R}$ and any integer $n$ strictly greater than the Krull dimension of $X$, we obtain a bijection between the Zariski cohomology groups $H^n_{\operatorname{Zar}}(X)$ with coefficients in the sheaf $I^n$ associated to the $n$th power of the fundamental ideal in the Witt ring $W(X)$ and the singular cohomology groups $H^\ast_{\operatorname{sing}}(X(\mathbb{R}), \mathbb{Z})$.

1. Introduction

Let $X$ be an algebraic variety over the field of real numbers and let $d$ denote the Krull dimension of $X$. Let $H^n$ denote the Zariski sheaf associated to the presheaf $U \mapsto H^0_{\acute{e}t}(U, \mathbb{Z}/2)$, where $H^0_{\acute{e}t}(U, \mathbb{Z}/2)$ denotes the étale cohomology of $U$ with $\mathbb{Z}/2\mathbb{Z}$-coefficients. Under the hypotheses that $X$ is smooth, integral, and quasi-projective a classic theorem of Raman Parimala and Jean-Louis Colliot-Thélène states that the sections of $H^n$ are in bijection with $H^0_{\operatorname{sing}}(X(\mathbb{R}), \mathbb{Z}/2)$ when $n \geq d + 1$ [7, Theorem 2.3.1]; it follows from this that there is a bijection of cohomology groups

$$H_{\operatorname{Zar}}^n(X, H^n) \simeq H^\ast_{\operatorname{sing}}(X(\mathbb{R}), \mathbb{Z}/2)$$

when $n \geq d + 1$, where $X(\mathbb{R})$ denotes the real points of $X$ equipped with the Euclidean topology and $H^\ast_{\operatorname{sing}}(X(\mathbb{R}), \mathbb{Z}/2)$ denotes the singular cohomology groups of the real points with $\mathbb{Z}/2\mathbb{Z}$-coefficients.

Let $W(X)$ denote the Witt ring of symmetric bilinear forms over $X$ and $I^n(X)$ the powers of the fundamental ideal c.f. [19]. Let $\mathcal{I}^n$ denote the Zariski sheaf associated to the presheaf $U \mapsto I^n(U)$. Let $\mathcal{I}$ denote the sheaf associated to the
presheaf $U \mapsto \mathcal{I}^n(U)/\mathcal{I}^{n+1}(U)$. The short exact sequence of sheaves

$$0 \to \mathcal{I}^{n+1} \to \mathcal{I}^n \to \mathcal{I}^n → 0$$

induces a long exact sequence in Zariski cohomology

$$\cdots \to H^n_{\text{Zar}}(X, \mathcal{I}^{n+1}) \to H^n_{\text{Zar}}(X, \mathcal{I}^n) \to H^n_{\text{Zar}}(X, \mathcal{I}^n) \xrightarrow{\partial} H^{n+1}_{\text{Zar}}(X, \mathcal{I}^{n+1}) \to \cdots$$

In the introduction to [11] Jean Fasel made the following assertions: the Zariski cohomology groups $H^n_{\text{Zar}}(X, \mathcal{I}^n)$ are the analogue of the singular cohomology groups $H^n_{\text{sing}}(X, \mathbb{R}, \mathbb{Z})$, while $H^n_{\text{Zar}}(X, \mathcal{I}^n)$ are the analogue of $H^n_{\text{sing}}(X, \mathbb{R}, \mathbb{Z}/2)$; the map $H^n_{\text{Zar}}(X, \mathcal{I}^{n+1}) \to H^n_{\text{Zar}}(X, \mathcal{I}^n)$ corresponds to the homomorphism $H^n_{\text{sing}}(X, \mathbb{R}, \mathbb{Z}) \xrightarrow{\beta} H^1_{\text{sing}}(X, \mathbb{R}, \mathbb{Z})$ induced by the multiplication by 2 on the coefficients; the connecting homomorphism $H^n_{\text{Zar}}(X, \mathcal{I}^n) \xrightarrow{\delta} H^n_{\text{Zar}}(X, \mathcal{I}^{n+1})$ is analogous to the Bockstein homomorphism $H^n_{\text{sing}}(X, \mathbb{R}, \mathbb{Z}/2) \xrightarrow{\delta} H^{n+1}_{\text{sing}}(X, \mathbb{R}, \mathbb{Z})$. Under the additional hypothesis that $X$ is affine, smooth, and has trivial canonical sheaf, he proved that $H^n_{\text{Zar}}(X, \mathcal{I}^n)$ and (3) by simply replacing everywhere $\mathbb{Z}/2$ with $\mathbb{Z}$ because when $n > d$ the étale cohomology groups $H^n_{\text{et}}(U, \mathbb{Z})$ are always torsion for any open subscheme $U$ of $X$ [25, Corollary 19.7.3.1].
Here, we obtain integral versions by demonstrating in Theorem (8.5) that for any scheme $X$ with two invertible in its global sections the signature induces an isomorphism of sheaves $\lim_{\to} I^n \to \text{supp}_* \mathbb{Z}$ which induces an isomorphism of cohomology groups

$$H^m_{\text{Zar}}(X, \lim_{\to} I^n) \overset{\text{sign}}{\cong} H^m(X, \mathbb{Z})$$

for all $m \geq 0$, where $\lim_{\to} I^n$ denotes the Zariski sheaf on $X$ obtained by taking the colimit of the system of sheaves

$$W_{\langle\langle -1 \rangle\rangle} \to I_{\langle\langle -1 \rangle\rangle} \to I_{\langle\langle 1 \rangle\rangle} \to \cdots$$

and $I_{\langle\langle -1 \rangle\rangle} \to I_{\langle\langle 1 \rangle\rangle}$ denotes the map induced by tensor product with the Pfister form $\langle\langle -1 \rangle\rangle := \langle 1, 1 \rangle$.

These global results follow from the local case, that is, the statement on the localization of the graded ring $I^*(A)$ from the abstract. Another way of stating this is to say that

$$\text{sign} : \lim_{\to} I^n(A) \to C(\text{sper} A, \mathbb{Z})$$

is bijective for any local ring $A$ with 2 invertible. Injectivity of (4) is well-known and follows from the local ring version of Pfister’s local-global principal (for instance [19, Ch. II §5] or directly in terms of the signature used in this article [21, Théorème 2.1 and Corollaire]). The statement that (4) is surjective is stronger than Mahé’s theorem, which states that the cokernel of $\text{sign} : W(A) \to C(\text{sper} A, \mathbb{Z})$ is 2-primary torsion for any commutative ring with 2 invertible. We believe it is known as well, but we don’t know of a reference in the literature for surjectivity of (4) when $A$ is local. We give a proof of bijectivity of (4) in Proposition (7.1) in a much different way using cohomological methods. For instance, in Theorem (5.2) we prove the Gersten conjecture for the Witt groups with 2 inverted of any regular excellent local ring. From this we deduce injectivity of (4) for any local ring with 2 invertible using “Hoobler’s trick”. Similarly, in Proposition (6.1) we prove a purity result for $\lim_{\to} I^n(A)$ in “geometric” cases and deduce surjectivity in general from this.

2. Total signature

For now and throughout this section, let $F$ be a field of characteristic different from two, although the hypothesis on the characteristic is not necessary for the definitions.

**Definition 2.1.** An ordering on $F$ is a subset $P \subset F$ satisfying the following:

1. $P + P \subset P$, $PP \subset P$;
2. $P \cap (-P) = 0$;
3. $P \cup -P = F$.

If $b - a \in P$, then one writes $a <_P b$. It follows from the axioms that if $F$ is nontrivial, then $1 >_P 0$. Also, one writes $\text{sgn}_P(a) = 1$ if $a \in P$ and $\text{sgn}_P(a) = -1$ if $a \in -P$. From the axioms one has that $\text{sgn}_P(ab) = \text{sgn}_P(a)\text{sgn}_P(b)$ for any $a, b \in$

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1A early version of Theorem (8.5) appeared in an Arxiv preprint of the author titled “From the global signature to higher signatures” [15, 4.4 Theorem]. The preprint version was significantly weaker in that it studied the “unramified” powers of the fundamental ideal instead of the actual powers of the fundamental ideal and it contained many unnecessary hypotheses. After improving and expanding upon the results of the preprint we split this part off as the separate article you see here.
$F^\times$, consequently assigning any $a \in F^\times$ to $\text{sgn}_P(a)$ determines a homomorphism $\text{sgn}_P : F^\times \to \{1, -1\}$ of groups. The pair $(F, P)$ is called an ordered field.

**Definition 2.2.** The real spectrum of $F$, denoted sper $F$, is the topological space formed by equipping the set of all orderings on $F$ with the topology generated by the subbasis consisting of subsets $H(a) \subset \text{sper} F$, $a \in F$, where $H(a)$ denotes the set of all orderings $P$ satisfying $a >_P 0$.

**Definition 2.3.** Let $P$ be an ordering on $F$. Any non-degenerate quadratic form $\phi$ over $F$ splits as an orthogonal sum $\phi \simeq \langle a_1 \rangle \perp \cdots \perp \langle a_n \rangle$ for some $a_1, \ldots, a_n \in F^\times$, then

$$\text{sign}_P([\phi]) := \sum_{i=1}^n \text{sgn}_P(a_i)$$

which is a homomorphism of rings [20, Chapter 1, Section 2, Satz 2]. Let $C(\text{sper} F, \mathbb{Z})$ denote the set of continuous integer valued functions on the real spectrum of $F$. The total signature is the ring homomorphism

$$\text{sign} : W(F) \to C(\text{sper} F, \mathbb{Z})$$

which assigns to an isometry class $[\phi]$ the continuous function $P \mapsto \text{sign}_P([\phi])$ [20, Chapter III, Section 8, Satz 1]. If $F$ has no ordering, then sign is trivial.

**Lemma 2.1.** Let $P$ be an ordering on $F$.

1. If $\phi$ is a diagonalizable form, $\phi \simeq \langle a_1 \rangle \perp \cdots \perp \langle a_n \rangle$ for some $a_1, \ldots, a_n \in F^\times$, then

$$\text{sign}_P([\phi]) := \sum_{i=1}^n \text{sgn}_P(a_i)$$

2. Let $a \in F^\times$. The Pfister form $\langle a \rangle := \langle 1, -a \rangle$ has total signature

$$\text{sign}([\langle a \rangle]) = 2^1 \{a < 0\}$$

3. Let $a_1, a_2, \ldots, a_n \in F^\times$. The $n$-fold Pfister form $\langle a_1, \ldots, a_n \rangle := \langle a_1 \rangle \otimes \cdots \otimes \langle a_n \rangle$ has total signature

$$\text{sign}([\langle a_1, \ldots, a_n \rangle]) = 2^n 2^1 \{a_1 < 0, \ldots, a_n < 0\}$$

**Proof.** For rank one forms $\langle a \rangle$, statement (1) follows from the definition of the signature. Since the signature is a ring homomorphism, statement (1) follows in general. To prove the equality in the statement (2) we use the definition and the
fact that the signature is a ring homomorphism:
\[
\text{sign}(\langle\langle a \rangle\rangle) = \text{sign}(\langle 1 - a \rangle) = \text{sign}(\langle 1 \rangle) - \text{sign}(\langle a \rangle)
\]
\[
= (1_{a>0} - 1_{a>0}) - (1_{a>0} - 1_{a<0})
\]
\[
= 1 - \mathbb{1}_{a>0} + \mathbb{1}_{a<0}
\]
\[
= \begin{cases} 0 & \text{if } a > 0 \\ 2 & \text{if } a < 0 \end{cases}
\]
\[
= 2\mathbb{1}_{a<0}
\]
To prove the third statement, use (2) and the fact that sign is a ring homomorphism.

**Definition 2.4.** As hyperbolic forms have even rank, assigning a quadratic form to its rank modulo 2 determines a ring homomorphism \(W(F) \to \mathbb{Z}/2\mathbb{Z}\). The kernel is denoted \(I(F)\) and is called the fundamental ideal of \(F\). The powers of the fundamental ideal \(I(F)\) are additively generated by Pfister forms \(\langle\langle a_1, \ldots, a_j \rangle\rangle\), so it follows from Lemma 2.1 that the signature induces a group homomorphism
\[
\text{sign} : I(F) \to \text{C(sper} F, 2^j \mathbb{Z})
\]
and the diagram below commutes
\[
\begin{array}{ccc}
I(F) & \xrightarrow{\text{sign}} & \text{C(sper} F, 2^j \mathbb{Z}) \\
\downarrow{\langle\langle -1 \rangle\rangle} & & \downarrow{2} \\
I^{j+1}(F) & \xrightarrow{\text{sign}} & \text{C(sper} F, 2^{j+1} \mathbb{Z})
\end{array}
\]
So after identifying
\[
\lim_{\rightarrow}(\text{C(sper} F, \mathbb{Z}))^2 \to \text{C(sper} F, 2\mathbb{Z})^2 \to \text{C(sper} F, 2^2 \mathbb{Z})^2 \to \cdots \cong \text{C(sper} F, \mathbb{Z})
\]
one obtains the map
\[
(5) \quad \lim_{\rightarrow}(W(F))^{\langle\langle -1 \rangle\rangle} \to I(F)^{\langle\langle -1 \rangle\rangle} \to I^2(F)^{\langle\langle -1 \rangle\rangle} \to \cdots \text{sign} \to \text{C(sper} F, \mathbb{Z})
\]
where \(\lim_{\rightarrow}\) denotes the colimit of the directed system of groups.

The following result first appeared in the paper of J. Arason and M. Knebusch cited in the Proposition below. Injectivity follows from A. Pfister’s local-global principal [24, Satz 22] and surjectivity follows immediately from the “Normality Theorem” of R. Elman and T.Y. Lam [9, 3.2].

**Proposition 2.5.** [1, Satz 2a.] The morphism (5) is a bijection.

3. Residues

Throughout this section \(A\) will denote a discrete valuation ring with fraction field \(K\) and residue field \(k = A/m\) of characteristic different from two. Let \(\pi\) be a uniformizing parameter for \(A\). The following lemma restates well-known facts on the second residue for Witt groups c.f. [23, Chap. IV (1.2)-(1.3)].

**Lemma 3.1.** (1) Every rank one quadratic form over \(K\) is isometric to some \(\langle c \rangle\), where \(c = bn^n\), \(b\) is a unit in \(A\), and either \(n = 0\) or \(n = 1\).
(2) The second residue $\partial_\pi : W(K) \to W(k)$ has the following description

\[
\partial_\pi((c)) = \begin{cases} 
\langle b \rangle & \text{if } n = 1 \\
0 & \text{if } n = 0
\end{cases}
\]

on rank one forms $\langle c \rangle$ as in (1).

(3) The second residue respects the powers of the fundamental ideal, that is, for any integer $n \geq 1$, it induces a homomorphism of groups

\[
\partial_\pi : I^n(K) \to I^{n-1}(k)
\]

where $I^0(k) := W(k)$.

**Definition 3.1.** Let $\pi$ be a uniformizing parameter for $A$. For any point $\xi \in \text{sper } A/m$, there are exactly two orderings on $K$ which induce the ordering of $\xi$ on $A/m$ and they differ by their sign on the uniformizing parameter [5, Theorem 10.1.10, see Definition 10.1.7 for induced orderings]. Denote them by $\eta_+$ and $\eta_-$, where $\text{sgn } \eta_+(\pi) = 1$ and $\text{sgn } \eta_-(\pi) = -1$. The group homomorphism

\[
\beta_\pi : C(\text{sper } K, \mathbb{Z}) \to C(\text{sper } A/m, \mathbb{Z})
\]

is defined by assigning $s \in C(\text{sper } K, \mathbb{Z})$ to the map $\xi \mapsto \beta_\pi(s)(\xi)$, where $\beta_\pi(s)(\xi) := s(\eta_+) - s(\eta_-)$. If $\text{sper } A/m = \emptyset$, then it is defined to be zero.

**Lemma 3.2.** Let $\pi$ be a uniformizing parameter for $A$. The morphism $\beta_\pi$ of Definition (3.1) has the following description on elements $\text{sign } (\langle c \rangle)$, where $c = b\pi^n$, $b$ is a unit in $A$, and either $n = 0$ or $n = 1$:

\[
\beta_\pi(\text{sign } (\langle c \rangle)) = \begin{cases} 
2\text{sign } (\langle b \rangle) & \text{if } n = 1 \\
0 & \text{if } n = 0
\end{cases}
\]

Proof. Let $c = b\pi^n$, where $b$ is a unit in $A$, and either $n = 0$ or $n = 1$. We have the following equalities which will prove the lemma. For any $\xi \in \text{sper } A/m$:

\[
\beta_\pi(\text{sign } (\langle c \rangle))(\xi) = \text{sign } \eta_+(\langle c \rangle) - \text{sign } \eta_-(\langle c \rangle)
\]

\[
= \text{sgn } \eta_+(c) - \text{sgn } \eta_-(c)
\]

\[
= \begin{cases} 
\text{sgn } \xi(\pi) - \text{sgn } \xi(\pi) & \text{if } n = 0 \text{ (both orderings induce } \xi) \\
\text{sgn } \eta_+(b\pi) - \text{sgn } \eta_-(b\pi) & \text{if } n = 1
\end{cases}
\]

\[
= \begin{cases} 
0 & \text{if } n = 0 \\
\text{sgn } \eta_+(b)\text{sgn } \eta_+(\pi) - \text{sgn } \eta_-(b)\text{sgn } \eta_-(\pi) & \text{if } n = 1
\end{cases}
\]

\[
= \begin{cases} 
0 & \text{if } n = 0 \\
\text{sgn } \eta_+(b) + \text{sgn } \eta_-(b) & \text{if } n = 1 \text{ (by definition of } \eta_+ \text{ and } \eta_-)
\end{cases}
\]

\[
= \begin{cases} 
0 & \text{if } n = 0 \\
\text{sgn } \xi(\langle b \rangle) + \text{sgn } \xi(\langle b \rangle) & \text{if } n = 1 \text{ (both orderings induce } \xi)
\end{cases}
\]

\[
= \begin{cases} 
0 & \text{if } n = 0 \\
2\text{sgn } \xi(\langle b \rangle) & \text{if } n = 1
\end{cases}
\]

\[\square\]

The next lemma follows from Lemmas (3.1) and (3.2).
Lemma 3.3. The diagram of abelian groups below is commutative.

\[
\begin{array}{ccc}
\lim_{n \to -1} I^n(K) & \xrightarrow{\partial} & \lim_{n \geq -1} I^n(k) \\
\text{sign} & & 2\text{sign} \\
C(\text{sper } K, \mathbb{Z}) & \xrightarrow{\beta_n} & C(\text{sper } k, \mathbb{Z})
\end{array}
\]

where \( \lim_{n \to -1} I^n(k) \) denotes the colimit over

\[
W(k) \xrightarrow{(\langle 1 \rangle)} W(k) \xrightarrow{(\langle -1 \rangle)} I(k) \xrightarrow{(\langle -1 \rangle)} I^2(k) \xrightarrow{(\langle -1 \rangle)} \ldots
\]

4. Real cohomology

In [25], C. Scheiderer developed a theory of real cohomology for schemes. It “globalizes” to schemes the singular cohomology of the real points of a real variety in the same way that étale cohomology globalizes the singular cohomology of the complex points of a complex variety. We recall the definition and some properties we will need following [25].

Definition 4.1. The real spectrum of a ring \( A \) is a topological space denoted by \( \text{sper } A \). As a set it consists of all pairs \( \xi = (p, P) \) with \( p \in \text{spec } A \) and \( P \) an ordering of the residue field \( k(p) \). For any point \( \xi \in \text{sper } A \), let \( k(\xi) \) denote the real closure of the ordered field \( k(p) \) with respect to \( P \). For \( a \in A \), write \( a(\xi) > 0 \) to indicate that the image of \( a \) in \( k(\xi) \) is positive. The sets of the form \( D(a) := \{ \xi \in \text{sper } A : a(\xi) > 0 \} \), \( a \in A \) a subsbasis for the topology on \( \text{sper } A \).

The real spectrum of a scheme \( X \) is the topological space \( X_r \) formed by gluing the real spectra of its open affine subschemes. This does not depend on the open cover of \( X \) that was chosen.

Definition 4.2. Let \( X \) be a scheme. First we recall the definition of the real site of \( X \), which we will also denote by \( X_r \). It is the category \( \text{O}(X_r) \) of open subsets of \( X_r \) equipped with the “usual” coverings, i.e. a family of open subspaces \( \{ U_\lambda \to U \} \) is a covering of \( U \in \text{O}(X_r) \) if \( U = \cup U_\lambda \). The category of sheaves of abelian groups on \( X_r \) is denoted \( \text{Ab}(X_r) \) and the category of abelian groups by \( \text{Ab} \). For any \( \mathcal{F} \in \text{Ab}(X_r) \), the real cohomology groups of \( X \) with coefficients in \( \mathcal{F} \) are the right derived functors of the global sections functor \( \Gamma : \text{Ab}(X_r) \to \text{Ab} \). They are denoted by

\[
H^p(X_r, \mathcal{F}) := R^p\Gamma \mathcal{F}
\]

where \( R^p\Gamma \) is the \( p \)-th derived functor of \( \Gamma \). When \( X = \text{spec } A \) is affine, we may write \( H^p(\text{sper } A, \mathcal{F}) \) instead of \( H^p(X_r, \mathcal{F}) \). For any abelian group \( M \), we will also denote by \( M \) the sheaf on \( X_r \) associated to the presheaf \( U \mapsto M \), \( U \) any open in \( X_r \). Such a sheaf is called a constant sheaf. Moreover, when the group \( M \) is equipped with the discrete topology we may write \( C(\text{sper } A, M) \) instead of \( H^0(\text{sper } A, M) \).

Remark 4.3. For any separated scheme of finite type over \( \mathbb{R} \) (or over any real closed field, with semi-algebraic cohomology replacing singular cohomology), the category of sheaves on \( X(\mathbb{R}) \), where \( X(\mathbb{R}) \) is equipped with the Euclidean topology, is equivalent to the category of sheaves on \( X \). [25, 15.4.1]. Let \( M \) be an abelian group. Since the sheaf cohomology of \( X(\mathbb{R}) \) with coefficients \( M \) coincides with the singular cohomology with coefficients \( M \) [25, 15.1 and 15.3.2], it follows that the
real cohomology groups \( H^p(X_r, M) \) coincide with the singular cohomology groups \( H^p(X(\mathbb{R}), M) \). Hence, they are finitely generated groups.

**Definition 4.4.** The assignment \((p, P) \mapsto \mathfrak{p}\) defines a continuous map of topological spaces from \(\text{sper}A \to \text{spec}A\). It is called the support map and is denoted by \(\text{supp}\). Similarly, we have for any scheme \(X\) a support map \(X_r \to X\). Let \(\text{Ab}(X_{\text{Zar}})\) denote the category of sheaves of abelian groups on the Zariski site \(X_{\text{Zar}}\). Since the support map is a continuous map of topological spaces it induces the direct image functor

\[
\text{supp}_* : \text{Ab}(X_r) \to \text{Ab}(X_{\text{Zar}})
\]

which is faithful and exact [25, Theorem 19.2].

**Lemma 4.1.** Let \(X\) be a scheme. For any sheaf \(F \in \text{Ab}(X_r)\),

\[
H^p(X_r, F) \simeq H^p_{\text{Zar}}(X, \text{supp}_* F)
\]

**Proof.** Using the Grothendieck spectral sequence for the composition of the functors \(\text{supp}_*\) and the global sections functor \(\Gamma\) we obtain a spectral sequence with \(E_2^{pq} = H^p_{\text{Zar}}(X, R^q\text{supp}_* F)\) that abuts to \(H^{p+q}(X_r, F)\). For \(q > 0\), the sheaves \(R^q\text{supp}_* F\) vanish [25, Theorem 19.2]. Therefore the edge maps in this spectral sequence determine isomorphisms \(H^p(X_r, F) \xrightarrow{\simeq} H^p_{\text{Zar}}(X, \text{supp}_* F)\) for \(p \geq 0\). \(\square\)

Next we recall the work of C. Scheiderer [26] in which he constructs a “Bloch-Ogus” style complex that computes real cohomology. The codimension of support filtration on \(X\) determines a spectral sequence abutting to real cohomology. Scheiderer shows that for regular excellent schemes the \(E_1\)-page is zero except for the complex \(E_1^{0,1}\) and hence obtains the result below. Recall that a locally noetherian scheme is called excellent if \(X\) can be covered by open affine subschemes spec \(A_\alpha\) where the \(A_\alpha\) are excellent rings [13, 7.8.5]. For a point \(x \in X\) of a scheme, we will denote \(\text{sper} k(x)\) by \(x_r\).

**Proposition 4.5.** [26, 2.1 Theorem] Let \(X\) be a regular excellent scheme. Let \(W\) be an open constructible subset of \(X_r\), and let \(F\) be a locally constant sheaf on \(W\). Then there is a complex of abelian groups

\[
\bigoplus_{x \in X^{(0)}} H^0_x(W,F) \to \bigoplus_{x \in X^{(1)}} H^1_x(W,F) \to \bigoplus_{x \in X^{(2)}} H^2_x(W,F) \to \cdots,
\]

natural in \(W\) and \(F\), whose \(q\)th cohomology group is canonically isomorphic to \(H^q(W,F), q \geq 0\). Here \(H^q_x(W,F) := H^q_{x_r \cap W}(\text{sper} \mathcal{O}_{X,x} \cap W, F)\). This complex is contravariantly functorial for flat morphisms of schemes.

The following Lemma is based on the proof of [26, 2.6 Proposition] where \(M = \mathbb{Z}/2\mathbb{Z}\).

**Lemma 4.2.** Let \(X\) be a regular excellent scheme which is integral with function field \(K\). Let \(x \in X^{(1)}\) and let \(\pi\) denote a choice of uniformizing parameter for \(\mathcal{O}_{X,x}\). Fix an integer \(n \geq 0\) and let \(M\) denote the constant sheaf \(\mathbb{Z}\). Denote by \(\partial\) the map

\[
H^0(\text{sper} K, M) \to H^1_x(\text{sper} \mathcal{O}_{X,x}, M)
\]

induced by first differential of the complex (6) from Proposition (4.5). Then, there is an isomorphism \(\iota_x : H^1_x(\text{sper} \mathcal{O}_{X,x}, M) \to H^0(x_r, M)\) for which \(\iota_x(\partial) = \beta_x\), where \(\beta_x\) is the map of Definition (3.1).
Proof. Let \( X' = \text{sper} \mathcal{O}_{X,x} \), \( Z' = x_r \), let \( i : Z' \to X' \) denote the inclusion, and let \( j : \text{sper} K \to X' \) denote the inclusion of the complement to \( Z' \). For any abelian sheaf \( M \) on \( X' \) the sequence

\[
M \to j_* j^* M \to i_* R^1 \tilde{i}^!(M) \to 0
\]

is exact (e.g. [27, 8.2.4 Corollary, Chapter II]). By [26, Lemma 1.3], for any locally constant sheaf \( M \) on \( X' \) the sequence

\[
M \to j_* j^* M \to i_* i^* M \to 0
\]

is exact, where \( \beta \) is defined on stalks \( \zeta \in Z' \) as \( (\beta_s)_\zeta = s(\eta_+) - s(\eta_-) \in M \). Hence we get an isomorphism \( \iota \pi \) of cokernels and a commutative diagram

\[
\begin{array}{ccc}
j_* j^* M(X') & \xrightarrow{\beta} & i_* i^* M(X') \\
\downarrow & & \downarrow \\
i_* i^* M(X') & &
\end{array}
\]

Tracking down all the definitions, one finds that Diagram (7) is equal to the diagram below.

\[
\begin{array}{ccc}
H^0(X' - Z', M) & \xrightarrow{\beta} & H^1_Z(X', M) \\
\downarrow & & \downarrow \\
H^0(Z', M) & &
\end{array}
\]

where the vertical map is the isomorphism \( \iota \pi \) chosen, the diagonal map is the map \( \beta \pi \) of Definition 3.1, and \( \text{sper} K \) equals \( X' - Z' \). This finishes the proof of the lemma. \( \square \)

Lemma 4.3. Let \( A \) be a regular excellent local ring with fraction field \( K \). Let \( X = \text{spec} A \), and for any \( x \in X^{(1)} \), let \( \pi_x \) be a choice of uniformizing parameter for \( \mathcal{O}_{X,x} \). Then, the sequence below is exact

\[
0 \to C(\text{sper} A, Z) \to C(\text{sper} K, Z) \oplus \bigoplus_{x \in X^{(1)}} C(\text{sper} k(x), Z)
\]

where \( \beta \pi \) is the map of Definition 3.1.

Proof. To prove the Lemma, choose isomorphisms \( \iota \pi \) for each \( x \in X^{(1)} \) as in Lemma (4.2) and then use Proposition (4.5). \( \square \)

5. On the Gersten conjecture with 2 inverted

Definition 5.1. Let \( A \) be a regular local ring with 2 invertible and let \( X = \text{spec} A \). We will work with the Gersten complex for the (derived) Witt groups of \( X \) as found for instance in [3, Introduction]. Recall that, after choosing isomorphisms \( \iota \) in order to identify the complex \( E_{1}^{*,d} \) of the coniveau spectral sequence with \( C(A,W) \), it may be written

\[
C(A,W) := \bigoplus_{x \in X^{(0)}} W(k(x)) \xrightarrow{\partial} \bigoplus_{x \in X^{(1)}} W(k(x)) \xrightarrow{\partial} \ldots \xrightarrow{\partial} \bigoplus_{x \in X^{(d)}} W(k(x))
\]

where \( d \) denotes the Krull dimension of \( A \). The differentials \( \partial \) may differ for different choices of isomorphisms \( \iota \) but the resulting complexes will all be isomorphic. One
may show that the differentials ∂ respect the filtration by powers of the fundamental ideal, for instance, this was shown by S. Gille [12, Corollary 7.3] for coherent Witt groups which gives the same complex since \( A \) is regular [3, Section 3, Another Construction]). Hence one obtains a subcomplex

\[
C(A, I^n) := \bigoplus_{x \in X^{(0)}} I^n (k(x)) \xrightarrow{d} \bigoplus_{x \in X^{(1)}} I^{n-1} (k(x)) \xrightarrow{d} \ldots \xrightarrow{d} \bigoplus_{x \in X^{(d)}} I^{n-d} (k(x))
\]

where we set \( I^n (k(x)) = W(k(x)) \) when \( m \leq 0 \). Define

\[
C(A, W/I^n) := C(A, W)/C(A, I^n)
\]
to be the quotient complex. The exact sequence of complexes

\[
0 \longrightarrow C(A, I^n) \longrightarrow C(A, W) \longrightarrow C(A, W/I^n) \longrightarrow 0
\]

determines a exact sequence of colimits

\[
(8) \quad 0 \longrightarrow C(A, \lim_n I^n) \longrightarrow C(A, \lim_n W) \longrightarrow C(A, \lim_n W/I^n) \longrightarrow 0
\]

where we define \( C(A, \lim_n I^n) := \lim C(A, I^n) \), \( C(A, \lim_n W/I^n) := \lim C(A, W/I^n) \), and \( C(A, W[\frac{1}{2}]) := \lim C(A, W) \xrightarrow{2} C(A, W) \xrightarrow{2} C(A, W) \xrightarrow{2} \ldots \).

**Theorem 5.2.** If \( A \) is a regular excellent local ring with 2 invertible, then the Gersten complex \( C(A, W[\frac{1}{2}]) \) is exact and \( H^n(C(A, W[\frac{1}{2}])) = W(A)[\frac{1}{2}] \).

**Proof.** We proceed by induction on the Krull dimension of \( A \). The Gersten complex without inverting 2 is exact already in low dimensions for any regular local ring [3, Lemma 3.2]. Fix \( A \) and assume the statement of the proposition is known for regular excellent local rings of Krull dimension less than that of \( A \). It is sufficient to show that the cohomology of \( C(A, W[\frac{1}{2}]) \) vanishes in degrees 2 and higher: one may use the coniveau spectral sequence with 2 inverted for Witt groups to show that this implies \( H^n(C(A, W[\frac{1}{2}])) = 0 \) in positive degree and \( H^0(C(A, W[\frac{1}{2}])) = W(A)[\frac{1}{2}] \), e.g. [3, Lemma 3.2]. For any regular parameter \( f \in A \), there is a short exact sequence of complexes

\[
0 \rightarrow C(A, W) \rightarrow C(A_f, W) \rightarrow C(A/f, W)[-1] \rightarrow 0
\]

for instance see [3, Lemma 3.3 and proof of Theorem 4.4]. Taking colimits it remains exact. As \( \dim A/f \) is strictly less than \( \dim A \) and \( A/f \) is again regular and excellent we have that \( C(A/f, W[\frac{1}{2}])[-1] \) is exact. Hence it remains to see that \( C(A_f, W) \) is exact in degrees 2 and higher. Note that for any \( p \in \text{spec } A_f \), \( \dim(A_f)_p \) is strictly less than \( \dim A \) and \( (A_f)_p \) is again regular and excellent, hence the cohomology of \( C(A_p, W[\frac{1}{2}]) \) agrees with \( H^*_\text{zar}(\text{spec } A_f, \lim W) \), where \( \lim W \) denotes the colimit over the sheaves

\[
W \xrightarrow{(-1)} W \xrightarrow{(-1)} W \xrightarrow{(-1)} \ldots
\]

For any point \( p \) in \( \text{spec } A_p \), using the induction hypothesis we have that the top row in the diagram below is exact and using Lemma (4.3) we have that the bottom
row is exact

\[ 0 \longrightarrow \lim_{\rightarrow} W((A_f)_p) \longrightarrow \lim_{\rightarrow} W(K) \oplus \bigoplus_{x \in Y(1)} \lim_{\rightarrow} W(k(x)) \]

\[ \downarrow \text{sign} \]

\[ \lim_{\rightarrow} W((A_f)_p) \longrightarrow C(\text{sper}(A_f), Z[\frac{1}{2}]) \oplus \bigoplus_{x \in Y(1)} C(\text{sper} k(x), Z[\frac{1}{2}]) \]

where \( Y := \text{spec}(A_f)_p \). Proposition (2.5) implies the middle vertical map is a bijection and the rightmost vertical map is an injection from which it follows that the leftmost vertical map is bijective. Thus we get an isomorphism \( \lim_{\rightarrow} W \cong \text{supp}^* Z[\frac{1}{2}] \) of sheaves on \( A_f \) as it is an isomorphism on stalks, where we use Lemma (4.1) to identify the sheaf \( \text{supp}^* Z[\frac{1}{2}] \). Then, the real cohomology groups \( H^*(\text{sper} A_f, Z[\frac{1}{2}]) \) are isomorphic to \( H^*_{\text{Zar}}(\text{spec} A_f, \lim_{\rightarrow} W) \), so it remains to prove their vanishing in degree 2 and higher. This is true since the real cohomology of local rings vanish in positive degree (in fact, semilocal too) \([25, \text{Proposition } (19.2.1)]\) and the real cohomology of \( \text{sper} A_f \) sits in a long exact sequence with that of \( \text{sper} A/f \) and \( \text{sper} A \) whenever \( A \) is regular excellent \([26, \text{Corollary } (1.10)]\). This finishes the proof. \( \square \)

Since the diagram below is commutative

\[ \lim_{\rightarrow} I^n(A) \longrightarrow W(A)[\frac{1}{2}] \]

\[ \downarrow \]

\[ \lim_{\rightarrow} I^n(K) \longrightarrow W(K)[\frac{1}{2}] \]

and the horizontal maps in the diagram are injective, we have the following Corollary to Theorem (5.2).

**Corollary 5.1.** Let \( A \) be a regular excellent local ring with 2 invertible. The map

\[ \lim_{\rightarrow} I^n(A) \rightarrow \lim_{\rightarrow} I^n(K) \]

is injective.

We will need the following result later.

**Lemma 5.1.** Let \( A \) be a regular excellent local ring with 2 invertible. The cohomology groups \( H^m(C(A, \lim_{\rightarrow} I^n)) \) vanish when \( m \geq 2 \).

**Proof.** Consider the long exact sequence in cohomology

\[ \cdots \rightarrow H^m(C(A, \lim_{\rightarrow} I^n)) \rightarrow H^m(C(A, \lim_{\rightarrow} W)) \rightarrow H^m(C(A, \lim_{\rightarrow} W/I^n)) \rightarrow \cdots \]

associated to the short exact sequence of complexes (8). The cohomology groups \( H^m(C(A, \lim_{\rightarrow} W)) \) vanish when \( m > 0 \) by Theorem (5.2). Then \( H^m(C(A, \lim_{\rightarrow} I^n)) \) is isomorphic to \( H^{m-1}(C(A, \lim_{\rightarrow} W/I^n)) \) for all \( m \geq 2 \). The cohomology groups \( H^m(C(A, \lim_{\rightarrow} W/I^n)) \) are two-primary torsion since the complex \( C(A, \lim_{\rightarrow} W/I^n) \) is, while the groups \( H^m(C(A, \lim_{\rightarrow} I^n)) \) have no 2-primary torsion since multiplication by 2

\[ C(A, \lim_{\rightarrow} I^n) \xrightarrow{2} C(A, \lim_{\rightarrow} I^n) \]

is an isomorphism of complexes. Thus both groups vanish proving the Lemma. \( \square \)
6. Purity of the limit in the local “geometric” case

For any prime $p$, we will use $\mathbb{Z}_{(p)}$ to denote the localization of $\mathbb{Z}$ at the prime ideal $(p) \in \text{spec } \mathbb{Z}$. In this section we prove purity of $\varprojlim I^n(A)$ in the case that $A$ is essentially smooth over either $\mathbb{Q}$ or $\mathbb{Z}_{(p)}$ (Proposition (6.1)). When $A$ is a local ring of mixed-characteristic $(0, p)$ with $p \neq 2$ (that is to say, the characteristic of the fraction field $K$ is 0 and the characteristic of the residue field is $p$) we will say that $A$ is essentially smooth over $\mathbb{Z}_{(p)}$, if $A = R_p$ is the localization at a prime $p \in \text{spec } R$ of a smooth and finite type $\mathbb{Z}_{(p)}$-algebra $R = \mathbb{Z}_{(p)}[T_1, T_2, \ldots, T_n]/I$.

**Lemma 6.1.** If $A$ is essentially smooth over $\mathbb{Z}_{(p)}$ for some prime $p \neq 2$ or over $\mathbb{Q}$, then the sequence

$$I^n(A)/I^{n+1}(A) \rightarrow I^n(K)/I^{n+1}(K) \overset{\partial_x}{\rightarrow} \bigoplus_{x \in X^{(1)}} I^{n-1}(k(x))/I^n(k(x))$$

is exact, where $X = \text{spec } A$.

**Proof.** Let $K_n^M(A)/2$ denote the “naive” Milnor $K$-theory defined exactly as for a field. Kummer theory gives a “symbol map” $K_n^M(A)/2 \rightarrow H^2_{\text{et}}(A, \mathbb{Z}/2)$ and in the commutative diagram

$$
\begin{array}{ccc}
K_n^M(A)/2 & \rightarrow & K_n^M(K)/2 \\
\downarrow & & \downarrow \\
0 & \rightarrow & H^2_{\text{et}}(A) \\
\downarrow & & \downarrow \\
I^n(A)/I^{n+1}(A) & \rightarrow & I^n(K)/I^{n+1}(K) \\
\downarrow & & \downarrow \\
\bigoplus_{x \in X^{(1)}} K_n^M(k(x))/2 & \rightarrow & \bigoplus_{x \in X^{(1)}} H^2_{\text{et}}(k(x))
\end{array}
$$

where $X = \text{spec } A$, the lower row is exact as a consequence of Gillet’s Gersten conjecture for étale cohomology in the $\mathbb{Z}_{(p)}$ case, and Bloch-Ogus in the $\mathbb{Q}$-case. Furthermore, the Galois symbol

$$K_n^M(A)/2 \rightarrow H^2_{\text{et}}(A)$$

is surjective when $A$ is essentially smooth over $\mathbb{Q}$ [17, 18] and when $A$ is essentially smooth over a discrete valuation ring\(^2\) c.f. [16, p.114, surjectivity of the Galois symbol]. Applying the Milnor conjecture as proved by V. Voevodsky, we have that the vertical maps in the middle and on the right are bijections. It follows that the upper row is exact in the middle. Since $\langle a, 1-a \rangle = 0$ in $W(A)$ for $a \in A^\times$ such that $1-a \in A^\times$, there is a well-defined homomorphism $K_n^M(A)/2 \rightarrow I^n(A)/I^{n+1}(A)$. Hence, in the commutative diagram

$$
\begin{array}{ccc}
K_n^M(A)/2 & \rightarrow & K_n^M(K)/2 \\
\downarrow & & \downarrow \\
I^n(A)/I^{n+1}(A) & \rightarrow & I^n(K)/I^{n+1}(K) \\
\downarrow & & \downarrow \\
\bigoplus_{x \in X^{(1)}} I^{n-1}(k(x))/I^n(k(x)) & \rightarrow & \bigoplus_{x \in X^{(1)}} I^n(k(x))/I^{n+1}(k(x))
\end{array}
$$

after using again using the Milnor conjecture, by which the vertical maps in the middle and on the right are bijections, we have that the lower row is exact in the middle. \[\square\]

**Lemma 6.2.** Let $A$ be essentially smooth over $\mathbb{Z}_{(p)}$, $p \neq 2$, or $\mathbb{Q}$.

\(^2\)In a correspondence with the author, B. Kahn explained that the passage from surjectivity in the essentially smooth over a field case to this case is easy and goes back to Lichtenbaum, if you grant Gillet’s Gersten conjecture for étale cohomology.
(1) There exists an integer $N$ such that $C(A, I^s) \xrightarrow{\sim} C(A, I^{s+1})$ is an isomorphism of complexes for all $s \geq N$.

(2) The groups $H^m(C(A, W))$ are $2^N$-torsion for all $m \geq 2$.

(3) There exists an integer $B \geq 0$ such that $2^B H^0(C(A, W)) \subset i_* (W(A))$, where $i^* : W(A) \to W(K)$ denotes the map induced by $i : \text{spec } K \to \text{spec } A$.

(4) $2^{B+N} H^0(C(A, W)) \subset i_* (I^N(A))$.

**Proof.** To prove (1), note that the cohomological 2-dimension of $k[x][\sqrt{-1}]$ is finite and bounded for all points $x$ by some $N$. Using the Arason-Pfister Haussatz and the Milnor conjecture for fields it follows that $I^N(k(x)[\sqrt{-1}])$ vanishes for all $x$ and from this it follows that for all $x$ we have an isomorphism $I^*(k(x)) \xrightarrow{\sim} I^{s+1}(k(x))$ for all $s \geq N$ [8, Corollary 35.27]. Hence $C(A, I^s) \xrightarrow{\sim} C(A, I^{s+1})$ is an isomorphism of complexes for all $s \geq N$. Hence $\lim_N C(A, I^n) \to C(A, I^N)$ is essentially smooth over either $\mathbb{Z}_p$, $p \neq 2$, or $\mathbb{Q}$. From the coniveau spectral sequence for Witt groups [4] we have a sequence

$W(A) \to E^{0,0} \to H^0(C(A, W))$

where $E^{0,0}$ consists of the elements in $H^0(C(A, W))$ which are mapped to zero under all the differentials in the spectral sequence leaving $H^0(C(A, W))$. So it suffices to show that some 2-power of $q$ maps to zero under all of the finitely many non-trivial differentials. The first non-trivial differential is $d : H^0(C(A, W)) \to H^2(C(A, W))$. Since $2^N H^2(C(A, W)) = 0$, we have that $d(2^n q) = 0$. Repeating this argument for each non-trivial differential $d : H^0(C(A, W)) \to H^{s+1}(C(A, W))$ we eventually find some 2-power $2^B$, which does not depend on $q$, such that $2^B q$ is in the kernel of all differentials, hence is in $E^{0,0}$. Finally, to prove (4), let $q \in 2^{B+N} H^0(C(A, I^N))$. Write it as $q = 2^{B+N} q_{\text{unr}}$ for some $q_{\text{unr}} \in H^0(C(A, W))$. By (3) we have that $2^B q_{\text{unr}} = i_*(Q)$ for some $Q \in W(A)$. So $i_*(2^N Q) = q$ where $2^N Q \in I^N(A)$. This proves $2^{B+N} H^0(C(A, I^N)) \subset i_*(I^N(A))$, finishing the proof of the Lemma. ∎

**Proposition 6.1.** Let $A$ be essentially smooth over either $\mathbb{Z}_p$, $p \neq 2$, or $\mathbb{Q}$. The sequence

$\lim_{\to} I^0(A) \to \lim_{\to} I^0(K) \xrightarrow{\oplus q_j} \bigoplus_{x \in X^{(1)}} \lim_{n \geq -1} I^n(k(x))$

is exact, where $\lim_{\to n \geq -1} I^n(k(x))$ denotes the colimit over $W(k(x)) \xrightarrow{(-1)} W(k(x)) \xrightarrow{(-1)} I(k(x)) \xrightarrow{(-1)} I^2(k(x)) \xrightarrow{(-1)} \cdots$. 


Lemma 7.1. If the following questions involving cohomological invariants satisfying "rigidity" in the sense of [14] for passing from the smooth geometric case to the geometric case for many semilocal rings $Q$, we see that, by the two out of three lemma, it is sufficient to prove that $q - i_*(Q_N + Q_{N+1} + \cdots + Q_{B+N-1}) \in H^0(C(A, B+I))^\ast$. Since we are in the "stable" range we have that $H^0(C(A, B+I)) = 2^{B+N}H^0(C(A, W) \subset i_*(I^N(A))$, where we used Lemma 6.2 (4) to obtain the latter inclusion. Hence we have a $Q_N \in I^N(A)$ such that

$$q = i_*(Q_N + Q_{N+1} + \cdots + Q_{B+N-1} + Q_N)$$

where $Q_N + Q_{N+1} + \cdots + Q_{B+N-1} + Q_N \in I^N(A)$. This finishes the proof. \hfill \Box

7. ON THE SIGNATURE: LOCAL CASE

In this section we use "Hoobler’s trick" which is a method due to R. Hoobler [14] for passing from the smooth geometric case to the geometric case for many questions involving cohomological invariants satisfying “rigidity” in the sense of the following Lemma.

Lemma 7.1. If $B$ is a local ring and $(B, I)$ a henselian pair such that 2 is invertible in both $B$ and $B/I$, then for all integers $n \geq 0$, the homomorphisms of groups

$$I^n(B) \to I^n(B/I)$$

and

$$I^n(B)/I^{n+1}(B) \to I^n(B/I)/I^{n+1}(B/I)$$

induced by the surjection $B \to B/I$ are bijections.

Proof. Let $B$ a local ring and $(B, I)$ a henselian pair such that 2 is invertible in both $B$ and $B/I$. Considering the diagram

$$
\begin{array}{cccccc}
0 & \to & I^{n+1}(B) & \to & I^n(B) & \to & I^n(B)/I^{n+1}(B) & \to & 0 \\
& \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & I^{n+1}(B/I) & \to & I^n(B/I) & \to & I^n(B/I)/I^{n+1}(B/I) & \to & 0
\end{array}
$$

we see that, by the two out of three lemma, it is sufficient to prove that $I^n(B) \to I^n(B/I)$ is a bijection for all $n \geq 0$. To prove injectivity for all $n \geq 0$, note that as $I^n(B)$ is contained in $W(B)$, it suffices to prove that $W(B) \to W(B/I)$ is injective.

We claim that the assignment $b + I \mapsto b$ determines a well-defined map $(B/I)^\ast/(B/I)^{\ast 2} \to B^\ast/B^{\ast 2}$. This claim follows from rigidity for étale cohomology due to Strano and Gabber independently but one may also prove it directly from the definition of Henselian pair: let $b_1, b_2 \in B^\ast$ such that $b_1 + I = b_2 + I$; the polynomial $T^2 - \frac{a}{b^2}$ has image $T^2 - 1$ in $B/I[T]$; as $(B, I)$ is a henselian pair, from the factorization $T^2 - 1 = (T - 1)(T + 1)$ in $B/I[T]$ we obtain a factorization $T^2 - \frac{a}{b^2} = (T - a)(T + a)$ in $B[T]$, for some $a \in B$; hence $b_1 = a^2b_2$ for some $a \in B^\ast$, that is, $b_1 = b_2$ in $B^\ast/(B^\ast)^{\ast 2}$. The claim follows. Next recall that for any semilocal ring $A$, the Witt

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3The author learned this from a recent preprint of Stefan Gille titled *On quadratic forms over semilocal rings*. 

---
group \(W(A)\) is a quotient of the group ring \(\mathbb{Z}[A^\times/A^\times^2]\) modulo the set of relations \(R\) additively generated by \([1] + [-1]\) and all elements

\[
\sum_{i=1}^h [a_i] - \sum_{i=1}^h [b_i]
\]

satisfying

\[
\prod_{i=1}^h (a_i) \sim \prod_{i=1}^h (b_i)
\]

with \(h = 4\) [19, Ch. 2, §4, Theorem 2]. Hence, in the commutative diagram below the rows are exact. Thus we obtain a well-defined map of cokernels \(W(B) \rightarrow W(B)\) such that the composition \(W(B) \rightarrow W(B/I) \rightarrow W(B)\) is the identity. This proves the desired injectivity. The composition \(W(B/I) \rightarrow W(B) \rightarrow W(B/I)\) is the identity hence \(W(B) \rightarrow W(B/I)\) is surjective. To prove surjectivity of \(I^n(B) \rightarrow I^n(B/I)\) for all \(n \geq 0\), recall that \(I^n(B/I)\) is additively generated by Pfister forms \(\langle b_1, b_2, \ldots, b_n \rangle\) where \(b_1, b_2, \ldots, b_n\) are units in \(B/I\) [2, Ch. V, Section 1, Remark 1.3]. For any Pfister form \(\langle b_1, b_2, \ldots, b_n \rangle\) we may lift the \(b_i\) to units \(b_i\) of \(B\) to obtain an element \(\langle b_1, b_2, \ldots, b_n \rangle \in I^n(B)\) mapping to it, proving surjectivity of \(I^n(B) \rightarrow I^n(B/I)\) and finishing the proof of the lemma.

**Proposition 7.1.** If \(A\) is a local ring with \(2 \in A^\times\), then the signature map

\[
\lim_{\rightarrow} I^n(A) \rightarrow C(\text{sper} A, \mathbb{Z})
\]

is a bijection.

**Proof.** As both groups respect filtered colimits, it suffices to consider the case where \(A\) is a localization of a finite type \(\mathbb{Z}\)-algebra: any local ring may be written as a union of its finitely generated subrings \(A_0\); pulling back the maximal ideal of \(A\) over \(A_0 \rightarrow A\) yields a prime ideal \(p_0 \in \text{spec} A_0\); localizing the \(A_0\) with respect to these primes yields a directed system of local rings \(A_{p_0}\) and taking the direct limit yields \(A\). From now on we assume \(A = R_p\), where \(p \in \text{spec} R\) and \(R = \mathbb{Z}[T_1, T_2, \ldots, T_n]/I\) for some ideal \(I\). We obtain a henselian pair \((B, I)\) for \(A\) as follows: let \(s\) denote the quotient map \(\mathbb{Z}[T_1, T_2, \ldots, T_n] \rightarrow R\) and let \(B_0 := \mathbb{Z}[T_1, T_2, \ldots, T_n]_{s^{-1}(p)}\) and similarly \(I_0 := I_{s^{-1}(p)}\); let \(B\) denote the henselization of \(B_0\) along \(I_0\) and \(I := I_0B\). Recall, the henselization along \(I_0\) is obtained by taking the colimit over the directed category consisting of those étale \(B_0\)-algebras \(C\) having the property that \(B_0/I_0 \rightarrow C/I_0C\) is an isomorphism. The map that \(B_0 \rightarrow B\) induces on quotients \(A = B_0/I_0 \rightarrow B/I\) is an isomorphism of local rings. In the diagram below, the horizontal maps induced by the surjection \(B \rightarrow B/I \simeq A\)

\[
\begin{array}{ccc}
\lim_{\rightarrow} I^n(B) & \longrightarrow & \lim_{\rightarrow} I^n(A) \\
\downarrow \text{sign} & & \downarrow \text{sign} \\
C(\text{sper} B, \mathbb{Z}) & \longrightarrow & C(\text{sper} A, \mathbb{Z})
\end{array}
\]
are isomorphisms for the powers of the fundamental ideal (Lemma (7.1)) and for real cohomology\(^4\). Therefore it suffices to prove bijectivity for \(B\). We claim that the local ring \(B\) is a filtered colimit of local rings which are essentially smooth over either \(\mathbb{Z}_p\), \(p \neq 2\), or over \(\mathbb{Q}\). In order to prove the claim, first note that the pullback of \(s^{-1}(p) \in \text{spec } \mathbb{Z}[T_1, T_2, \cdots, T_n]\) over \(\mathbb{Z} \to \mathbb{Z}/[T_1, T_2, \cdots, T_n]\) yields a prime \(\langle p \rangle \in \text{spec } \mathbb{Z}\) and localizing with respect to this prime induces \(\mathbb{Z}_p(\langle p \rangle) \to B_0\). When \(\langle p \rangle = 0\) it follows that \(B_0\) contains \(\mathbb{Q}\), otherwise \(B_0\) contains \(\mathbb{Z}_p(\langle p \rangle)\). The morphisms \(\mathbb{Z}_p(\langle p \rangle) \to B_0\) and \(B_0 \to B\) are both flat with geometrically regular fibers, hence the composition \(\mathbb{Z}_p(\langle p \rangle) \to B\) has these properties. Then it follows from Popescu’s theorem that \(B\) is a filtered colimit of either smooth \(\mathbb{Z}_p\)-algebras or \(\mathbb{Q}\)-algebras \(A_\alpha\). Pulling back the maximal ideal over \(A_\alpha \to B\) and localizing one obtains the statement of the claim. Thus, we may assume that \(B\) is essentially smooth over \(\mathbb{Q}\) or \(\mathbb{Z}_p(\langle p \rangle)\). Then we map apply Lemma (4.3) to get exactness of the lower row in the diagram below

\[
\begin{array}{cccccc}
0 & \xrightarrow{\lim} & I^n(B) & \xrightarrow{\oplus_{\beta}} & \bigoplus_{x \in Y(1)} \lim_{n \to 2} I^n(k(x)) & \xrightarrow{\text{sign}} & C(\text{sper } B, \mathbb{Z}) \\
& & & \downarrow & & \downarrow & \\
0 & \xrightarrow{\lim} & C(\text{sper } K, \mathbb{Z}) & \xrightarrow{\oplus_{\beta}} & \bigoplus_{x \in Y(1)} C(\text{sper } k(x), \mathbb{Z}) & & \\
\end{array}
\]

where \(Y = \text{spec } B\). We have exactness of the upper row by Proposition (6.1) and Corollary (5.1). Using the bijection of Proposition (2.5) we get that the middle vertical map in the diagram above is bijective and the rightmost vertical map in injective. The square on the right commutes by Lemma (3.3), hence \(\lim I^n(B) \to C(\text{sper } B, \mathbb{Z})\) is bijective, finishing the proof of the theorem. \(\square\)

The following corollary is well-known as mentioned in the introduction.

**Corollary 7.1.** Let \(A\) be a local ring with \(2 \in A^\times\). Then the signature induces an isomorphism

\[W(A)[\frac{1}{2}] \xrightarrow{\sim} C(\text{sper } A, \mathbb{Z})[\frac{1}{2}]\]

Proof. From the preceding theorem we have that any \(f \in C(\text{sper } A, \mathbb{Z})\) has \(2^n f = \text{sign}(Q)\) for some \(Q \in I^n(A) \subset W(A)\), proving surjectivity, and that for any \(Q' \in W(A)\), if \(\text{sign}(Q') = 0\) then \(2^n Q' = 0\) for some \(n\), proving injectivity. \(\square\)

**Remark 7.2.** Let \(A = \bigoplus_{n \geq 0} A_n\) be a \(\mathbb{Z}_s\)-graded ring and let \(s \in A_1\) be a homogeneous element of degree 1. Recall that the homogeneous localization \(A(s)\) is the subring of degree zero elements in the localization of \(A\) with respect to \(\{1, s, s^2, \cdots\}\), and that \(A(s) \cong A/(s-1)A\) as rings. Furthermore, \(A(s)\), may be obtained by taking the direct limit of the sequence \(A_0 \xrightarrow{\delta} A_1 \xrightarrow{\delta} A_2 \xrightarrow{\delta} \cdots\)

**Corollary 7.2.** Let \(A\) be a local ring with 2 invertible.

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\(^4\)The following proof was communicated to the author by C. Scheiderer: every point in \(\text{sper } B\) specializes to a point in \(\text{sper } B/I\) by the henselian property; since any real spectrum is a ”normal” spectral space, meaning that every point of \(X_r\) specializes to a unique closed point, the restriction map in sheaf cohomology \(H^*(X_r, F) \to H^*(X_{r, \text{max}}, F)\) is an isomorphism for every sheaf \(F\) on any scheme \(X\); thus restriction gives isomorphisms \(H^*(\text{sper } (B), F) \to H^*(\text{sper } (B), F) \cong H^*(\text{sper } (B/1), F)\).
Corollary 7.3. Let \( \langle \langle -1 \rangle \rangle \) denote the homogeneous localization of the graded ring \( \bigoplus_{n \geq 0} I^n(A) \) with respect to the element \( \langle \langle -1 \rangle \rangle = (1,1) \in I(A) \). The signature defines an isomorphism of rings
\[
I^n(A)_{\langle \langle -1 \rangle \rangle} \simeq \text{C(sper } A, \mathbb{Z})
\]

(2) Let \( \overline{\mathcal{T}}(A)_{\langle \langle -1 \rangle \rangle} \) denote the homogeneous localization of the graded ring \( \bigoplus_{n \geq 0} \overline{\mathcal{T}}(A) \) with respect to \( \langle \langle -1 \rangle \rangle = (1,1) \in \overline{\mathcal{T}}(A) \), where \( \overline{\mathcal{T}}(A) := I^n(A)/I^{n+1}(A) \). The signature defines an isomorphism of rings
\[
\overline{\mathcal{T}}(A)_{\langle \langle -1 \rangle \rangle} \simeq \text{C(sper } A, \mathbb{Z}/2)
\]

Proof. The statement (1) is an immediate corollary to the preceding proposition in view of Remark (7.2). To prove (2), we obtain the desired isomorphism as an isomorphism of cokernels in the diagram below

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & \lim_{\longrightarrow n \geq 1} I^n(A) & \longrightarrow & \lim_{\longrightarrow n \geq 1} I^n(A) & \longrightarrow & \lim_{\longrightarrow n \geq 1} \overline{\mathcal{T}}(A) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \text{C(sper } A, 2\mathbb{Z}) & \longrightarrow & \text{C(sper } A, \mathbb{Z}) & \longrightarrow & \text{C(sper } A, \mathbb{Z}/2) & \longrightarrow & 0
\end{array}
\]

where \( \lim_{\longrightarrow n \geq 1} \text{C(sper } A, 2\mathbb{Z}) \) is an isomorphism since in the diagram

\[
\begin{array}{ccc}
\lim_{\longrightarrow n \geq 1} I^n(A) & \longrightarrow & \text{C(sper } A, 2\mathbb{Z}) \\
\langle \langle -1 \rangle \rangle & \downarrow & 2 \\
\lim_{\longrightarrow n \geq 1} I^n(A) & \longrightarrow & \text{C(sper } A, \mathbb{Z})
\end{array}
\]

the vertical maps are isomorphisms as is the lower horizontal map. \( \square \)

Corollary 7.3. Let \( A \) be a local ring with \( 2 \) invertible. Let \( H^*_\mathcal{A}(A, \mathbb{Z}/2)_{\langle \langle -1 \rangle \rangle} \) denote the homogeneous localization of the cohomology ring \( \bigoplus_{n \geq 0} H^\mathcal{A}_n(A, \mathbb{Z}/2\mathbb{Z}) \) with respect to \( \langle \langle -1 \rangle \rangle = (1,1) \in \overline{\mathcal{T}}(A) \). Then the \( n \)th cohomological invariant \( \tau_n : \overline{\mathcal{T}} \to H^\mathcal{A}_n(A, \mathbb{Z}/2\mathbb{Z}) \), which assigns the class of a Pfister form \( \langle \langle a_1, \cdots, a_n \rangle \rangle \) to the cup product \( (a_1) \cup \cdots \cup (a_n) \) determines a well-defined homomorphism
\[
\tau_* : \overline{\mathcal{T}}(A)_{\langle \langle -1 \rangle \rangle} \simeq H^\mathcal{A}_n(A, \mathbb{Z}/2)(-1)
\]

which is an isomorphism of rings.

Proof. For any local ring \( A \) essentially smooth over \( \mathbb{Z}_{(p)} \) or \( \mathbb{Q} \), the diagram commutes

\[
\begin{array}{cccccc}
I^n(A) & \longrightarrow & I^n(K) & \longrightarrow & \bigoplus_{x \in X(1)} I^{n-1}(k(x))/2 \\
\downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & H^\mathcal{A}_n(A, \mathbb{Z}/2) & \longrightarrow & H^\mathcal{A}_n(K) & \longrightarrow & \bigoplus_{x \in X(1)} H^\mathcal{A}_{n-1}(k(x))
\end{array}
\]

and the lower row is exact as the Gersten conjecture is known for étale cohomology in this case. As the diagram commutes it follows that \( I^n(A)/I^{n+1}(A) \) maps into \( H^n(A, \mathbb{Z}/2) \). Let \( \tau_n \) denote this map. As the lower row is exact, it has the description asserted on Pfister forms. Using rigidity and the fact that both groups respect filtered colimits as we did in the proof of Theorem (8.5) we obtain the map \( \tau_n \) for
any local ring, and after localizing, we obtain the map in the commutative diagram below

\[
\begin{array}{c}
\mathcal{F}(A)_{(-1)} \xrightarrow{\pi_*} H^*_\text{et}(A, \mathbb{Z}/2)(-1) \\
\cong \downarrow \cong \\
\text{C(sper } A, \mathbb{Z}/2) \\
\end{array}
\]

where we use that for any semi-local ring \( A \) with 2 invertible the signature modulo 2 defines an isomorphism

\[
H^*_\text{et}(A, \mathbb{Z}/2)(-1) \cong \text{C(sper } A, \mathbb{Z}/2)
\]

of rings. This is due to J. Burési and L. Mahé in the semi-local case [22, 6] and C. Scheiderer in general [25, Corollary (7.10.3) and (7.19)]. From the isomorphisms in the diagram, the desired isomorphism follows. 

8. Globalization

In this section \( X \) will always denote a scheme. Let \( W(X) \) denote the Witt ring of symmetric bilinear forms over \( X \) c.f. [19].

**Definition 8.1.** Recall that the **global signature** is the ring homomorphism

\[ \text{sign} : W(X) \rightarrow H^0(X_r, \mathbb{Z}) \]

that assigns an isometry class \([\phi]\) of a symmetric bilinear form \( \phi \) over \( X \) to the function on \( X_r \) defined by

\[ \text{sign}([\phi])(x, P) := \text{sign}_P([i^*_x \phi]) \]

where \( i_x : x \rightarrow X \) is any point and \( P \) is any ordering on \( k(x) \) c.f. [21].

**Definition 8.2.** There is a well-defined ring homomorphism \( W(X) \rightarrow H^0_\text{et}(X, \mathbb{Z}/2\mathbb{Z}) \), called the **rank**, which assigns an isometry class of a symmetric bilinear form \([\mathcal{E}, \phi]\) over \( X \) to the rank of its underlying vector bundle \( \mathcal{E} \) modulo 2 c.f. [19, Chapter 1, §7]. The kernel of the rank map is called the **fundamental ideal** and is denoted by \( I(X) \).

It follows from the definitions that the diagram below commutes.

\[
\begin{array}{c}
W(X) \xrightarrow{\text{sign}} H^0(X_r, \mathbb{Z}) \\
\downarrow \text{rank mod 2} \downarrow \downarrow \\
H^0_\text{et}(X, \mathbb{Z}/2\mathbb{Z}) \xrightarrow{h_0} H^0(X_r, \mathbb{Z}/2\mathbb{Z}) \\
\end{array}
\]

where \( h_0 \) denotes the **signature modulo 2** defined as follows: if \( \alpha \in H^0_\text{et}(X, \mathbb{Z}/2\mathbb{Z}) \) and \( \xi : x \rightarrow X \) is the inclusion of a “real” point, that is, for some \( (x, P) \in X_r \), then \( h_0(\alpha) \) evaluated at \( \xi \) is \( \xi^* \alpha \in H^0(\text{et}, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} \); write \( \alpha(\xi) \) for this element of \( \mathbb{Z}/2\mathbb{Z} \), so \( h_0(\alpha) \) is the locally constant map \( X_r \rightarrow \mathbb{Z}/2\mathbb{Z} \), \( \xi \mapsto \alpha(\xi) \) c.f. [25, (7.19.1)].

**Definition 8.3.** As there is an exact sequence

\[ 0 \rightarrow H^0(X_r, 2\mathbb{Z}) \rightarrow H^0(X_r, \mathbb{Z}) \rightarrow H^0(X_r, \mathbb{Z}/2\mathbb{Z}) \]

one finds using commutativity of Diagram (10) that the restriction of the signature to \( I(X) \) defines the homomorphism of groups below.

\[ I(X) \rightarrow H^0(X_r, 2\mathbb{Z}) \]
For \( n \geq 0 \), let \( I^n(X) \) denote the powers of the fundamental ideal and \( I^0(X) = W(X) \). Since the global signature is a ring homomorphism that maps elements of \( I(X) \) into \( H^0(X_r, 2\mathbb{Z}) \), it follows that for any \( n \geq 0 \) it induces a homomorphism

\[
I^n(X) \to H^0(X_r, 2^n\mathbb{Z})
\]

of groups. Moreover, multiplication by \( 2 = \langle\langle -1 \rangle\rangle \in I(X) \) induces a homomorphism

\[
I^n(X) \langle\langle -1 \rangle\rangle \to I^{n+1}(X)
\]

such that the diagram below commutes.

\[
\begin{array}{ccc}
I^n(X) & \xrightarrow{\text{sign}} & H^0(X_r, 2^n\mathbb{Z}) \\
\downarrow^\langle\langle -1 \rangle\rangle & & \downarrow 2 \\
I^{n+1}(X) & \xrightarrow{\text{sign}} & H^0(X_r, 2^{n+1}\mathbb{Z})
\end{array}
\]

Hence, we obtain a homomorphism

\[
\lim_{\to} I^n(X) \to H^0(X_r, \mathbb{Z})
\]

where \( \lim_{\to} I^n(X) \) denotes the direct limit of the sequence \( W(X) \langle\langle -1 \rangle\rangle \to I(X) \langle\langle -1 \rangle\rangle \to I^2(X) \langle\langle -1 \rangle\rangle \to \cdots \) of groups.

**Definition 8.4.** It follows from Lemma (4.1) that \( \text{supp}^* \mathbb{Z} \) is the Zariski sheaf \( U \mapsto H^0(U_r, \mathbb{Z}) \) on \( X \). Recall that \( \mathcal{I}^n \) denotes the Zariski sheaf on \( X \) associated to the presheaf \( U \mapsto I^n(U) \). For any integer \( n \geq 0 \), the restriction of the global signature to the powers of the fundamental ideal of Definition (8.3) induces a homomorphism

\[
\mathcal{I}^n \to \text{supp}^* 2^n\mathbb{Z}
\]

of Zariski sheaves on \( X \). Similarly, \( I^n(X) \langle\langle -1 \rangle\rangle \to I^{n+1}(X) \langle\langle -1 \rangle\rangle \) induces a homomorphism \( \mathcal{I}^n \langle\langle -1 \rangle\rangle \to \mathcal{I}^{n+1} \langle\langle -1 \rangle\rangle \) of sheaves for any \( n \geq 0 \), and a homomorphism of sheaves

\[
\lim_{\to} \mathcal{I}^n \to \text{supp}^* \mathbb{Z}
\]

where \( \lim_{\to} \mathcal{I}^n \) denotes the direct limit of the sequence \( \mathcal{W} \langle\langle -1 \rangle\rangle \to \mathcal{I} \langle\langle -1 \rangle\rangle \to \mathcal{I}^2 \langle\langle -1 \rangle\rangle \to \cdots \). Similarly, the signature induces a morphism of sheaves

\[
\mathcal{W}[\frac{1}{2}] \to \text{supp}^* \mathbb{Z}[\frac{1}{2}]
\]

where \( \mathcal{W}[\frac{1}{2}] \) is the sheaf associated to the presheaf \( U \mapsto W(U)[\frac{1}{2}] \) and \( \text{supp}^* \mathbb{Z}[\frac{1}{2}] \) is the sheaf \( U \mapsto H^0(U_r, \mathbb{Z}[\frac{1}{2}]) \).

**Theorem 8.5.** Let \( X \) be a scheme with \( 2 \) invertible in its global sections.

1. The signature morphism of sheaves of Definition (8.4)

\[
\lim_{\to} \mathcal{I}^n \to \text{supp}^* \mathbb{Z}
\]

is an isomorphism.

2. The signature morphism of sheaves of Definition (8.4)

\[
\mathcal{W}[\frac{1}{2}] \to \text{supp}^* \mathbb{Z}[\frac{1}{2}]
\]

is an isomorphism.
The signature induces an isomorphism of short exact sequence of sheaves on $X$
\[ 0 \longrightarrow \lim \mathcal{I}^n \longrightarrow W[1/2] \longrightarrow \lim W/I^n \longrightarrow 0 \]
\[ 0 \longrightarrow \text{supp}^\ast Z \longrightarrow \text{supp}^\ast Z[1/2] \longrightarrow \text{supp}^\ast (Z[1/2]/Z) \longrightarrow 0 \]
where $W/I^n$ denotes the sheaf associated to the presheaf $U \mapsto W(U)/I^n(U)$.

(4) The signature induces an isomorphism of short exact sequence of sheaves on $X$
\[ 0 \longrightarrow \lim \mathcal{I}^{n\langle(-1)\rangle} \longrightarrow \lim \mathcal{I}^n \longrightarrow \lim \mathcal{I}^n/\langle\langle -1 \rangle\rangle \longrightarrow 0 \]
\[ 0 \longrightarrow \text{supp}^\ast Z \longrightarrow \text{supp}^\ast Z[1/2] \longrightarrow \text{supp}^\ast (Z[1/2]/Z) \longrightarrow 0 \]
where $I^n$ denotes the sheaf associated to the presheaf $U \mapsto I^n(U)/I^{n+1}(U)$.

**Proof.** Statements (1) and (2) follow immediately from the local case, Proposition (7.1) and Corollary (7.1) respectively, as it is sufficient to prove that they induce an isomorphism on stalks. As $\text{supp}^\ast$ is exact, statements (3) and (4) may be obtained by applying $\text{supp}^\ast$ to the analogous short exact sequences of groups and then using the two out of three lemma to conclude, but for (4), one should note that
\[
\lim \mathcal{I}^{n\langle(-1)\rangle} \longrightarrow \lim \mathcal{I}^n \\
\lim_{n \geq 1} I^n \]
is an isomorphism in order to obtain exactness of the top row of the diagram in (4). \qed

The next corollary is an immediate consequence of the previous theorem and Lemma (4.1).

**Corollary 8.1.** Let $X$ be a scheme with 2 invertible.

(1) For any $m \geq 0$, the morphism (11) induces an isomorphism of cohomology groups
\[ H^m_{Zar}(X, \lim \mathcal{I}^n) \rightarrow H^m(X_r, Z) \]
(2) For any $m \geq 0$, the morphism (12) induces an isomorphism of cohomology groups
\[ H^m_{Zar}(X, W[1/2]) \rightarrow H^m(X_r, Z[1/2]) \]

**Corollary 8.2.** Let $X$ be a scheme with 2 invertible which is quasi-separated and quasi-compact. Then, there is an isomorphism of cohomology groups for all $m \geq 0$
\[ \bigoplus_{m \geq 0} H^m_{Zar}(X, \lim \mathcal{I}^n) \simeq \lim H^m_{et}(X, Z/2) \]

**Proof.** Under the hypotheses stated C. Scheider has proved [25, Corollary (7.19)] that there is an isomorphism
\[ \lim H^m_{et}(X, Z/2) \simeq \bigoplus_{m \geq 0} H^m(X_r, Z/2) \]
and from Theorem 8.5 one has an isomorphism $H^{m}_{Zar}(X, \overline{I^n}) \xrightarrow{\sim} H^m(X_{\mathbb{R}}, \mathbb{Z}/2)$ for all $m \geq 0$. Thus one obtains the isomorphism stated. 

**Corollary 8.3.** If $X$ is a real variety, by which we mean a scheme which is separated and of finite type over $\mathbb{R}$, and the Krull dimension of $X$ is $d$, then whenever $n \geq d + 1$ the signature induces an isomorphism in cohomology $H^{m}_{Zar}(X, I^n) \xrightarrow{\sim} H^m_{sing}(X(\mathbb{R}), \mathbb{Z})$ for all integers $m \geq 0$ and an isomorphism of long exact sequences as stated in the introduction.

**Proof.** It suffices to see that the morphism of sheaves $I^n \langle\langle -1 \rangle\rangle \rightarrow I^{n+1}$ is an isomorphism for $n \geq d + 1$, for then multiplication by $2^{d+1}$ defines an isomorphism of sheaves $I^{d+1} \rightarrow \lim_{\rightarrow} I^n$ and hence we obtain the statement of the corollary using Theorem 8.5 in view of Remark (4.3). When $n \geq d + 1$, for any $U$ open in $X$ we have an isomorphism of kernels in the diagram of residues below

$$
\begin{array}{c}
0 \rightarrow I^n(U) \rightarrow I^n(K) \rightarrow \bigoplus_{x \in X(\mathbb{R})} I^{n-1}(k(x)) \\
\downarrow 2 \downarrow 2 \downarrow 2 \\
0 \rightarrow I^{n+1}(U) \rightarrow I^{n+1}(K) \rightarrow \bigoplus_{x \in X(\mathbb{R})} I^n(k(x))
\end{array}
$$

since the two rightmost vertical maps are isomorphisms for $n \geq d + 1$, which proves the desired isomorphism. 

**References**


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