Mathematics 115 – Fall ’10: Lab 7

EXPONENTIAL DECAY

In the exponential growth we considered a model of the type

\[ \frac{dP}{dt} = kP, \]

where \( k \) is a positive constant, called growth rate. In many applications, the constant \( k \) is actually the difference between a growth rate and a decay rate. In the population growth model, we have

\[ k = \text{(birthrate)} - \text{(deathrate)}. \]

If the death rate is bigger than the birthrate, than the equation (1) can be written as

\[ \frac{dP}{dt} = -kP, \]

where \( k \) is still positive, but now is given by \( k = \text{(deathrate)} - \text{(birthrate)} \). The solution to this equation is a decreasing exponential of the form

\[ P(t) = P_0e^{-kt}, \]

where \( P_0 \), as in the exponential growth, is the initial value of \( P(t) \).

NEWTON’S LAW OF COOLING

When an object is put in an environment which is at a lower temperature, its temperature starts to cool according to a mathematical model called Newton’s law of cooling.

Newton’s law of cooling: the rate of heat loss of a body is proportional to the difference in temperatures between the body and its surroundings.

In formulas, we can write

\[ \frac{dT}{dt} = -k(T - C) \]

where \( T \) is the temperature of the body and it’s a function of time \( t \), \( C \) is the temperature of the surrounding (which is considered constant) and \( k \) is a constant depending on the material of the body and the surrounding.
To find a solution to this differential equation, we consider another function $y(t) = T(t) - C$, which is the difference of temperature between the body and the surrounding. If we differentiate $y(t)$ with respect to time, we get

\[
\frac{dy}{dt} = \frac{dT}{dt} - \frac{dC}{dt} = -k(T - C) = -ky.
\]

Thus the new function $y(t)$ satisfies the (easier) differential equation

\[
y'(t) = -ky,
\]

whose solution is given by $y(t) = P_0 e^{-kt}$. If we substitute the expression for $y(t)$ we get

\[
T(t) - C = P_0 e^{-kt} \implies T(t) = C + P_0 e^{-kt}
\]

with $P_0$ constant to be determined based on the initial value of the temperature. In particular, substituting $t_0 = 0$ in the expression of $y(t)$, we find that $P_0$ is the difference between the temperature of the body and the temperature of the surrounding when $t = 0$.

**Exercise: Newton’s Law of Cooling**

The temperature of a hot liquid is 100. The liquid is placed in a refrigerator where the temperature is 40, and it cools to 90 in 5 min.

(a) Find the value of the constant $P_0$ in the Newton’s Law of Cooling.
(b) Find the value of the constant $k$.
(c) What is the temperature after 10 min?
(d) How long does it take the liquid to cool to 41?
(e) Find the rate of change of the temperature and interpret its meaning.