Linear Algebra - Math 221-002

Answers to even exercises

Section 1.1

- 1.1.10 \((-3, -5, 6, -3)\)
- 1.1.12 Inconsistent
- 1.1.14 \((2, -1, 1)\)
- 1.1.16 Consistent, infinitely many solutions.
- 1.1.20 Consistent for any \(h\). If \(h = -2\) infinitely many solutions, if \(h \neq -2\) one solution.
- 1.1.22 Consistent if \(3h + 5 = 0\), i.e. \(h = -5/3\).

Section 1.2

- 1.2.2 a: Reduced echelon form. b: Echelon form. c: Not echelon. d: Echelon form.
- 1.2.4
  \[
  \begin{bmatrix}
  1 & 0 & -1 & 0 \\
  0 & 1 & 2 & 0 \\
  0 & 0 & 0 & 1
  \end{bmatrix}
  \]
  Pivot columns: 1, 2, and 4. The associated system is inconsistent.
- 1.2.12
  \[
  \begin{cases}
  x_1 = 5 + 7x_2 - 6x_4 \\
  x_2 \text{ is free} \\
  x_3 = -3 + 2x_4 \\
  x_4 \text{ is free}
  \end{cases}
  \]
- 1.2.16 a: Unique solution. b: Consistent, with infinitely many solutions.
- 1.2.24 The system is inconsistent, because the pivot in column 5 means that there is a row of the form \([0 0 0 0 1]\). Since the matrix is the augmented matrix of a linear system, Theorem 2 shows that such system has no solution.
Section 1.3

- 1.3.8 \( w = -u + 2v, \quad x = -2u + 2v, \quad y = -2u + 3.5v, \quad z = -3u + 4v \)
- 1.3.10 \[
\begin{bmatrix}
4 \\
1 \\
8
\end{bmatrix} x_1 + \begin{bmatrix}
1 \\
-7 \\
6
\end{bmatrix} x_2 + \begin{bmatrix}
3 \\
-2 \\
-5
\end{bmatrix} x_3 = \begin{bmatrix} 9 \\
2 \\
15
\end{bmatrix}
\]
- 1.3.14 Yes, \( b \) is a linear combination of the columns of \( A \).
- 1.3.22 Construct any \( 3 \times 4 \) matrix in echelon form that corresponds to an inconsistent system. Perform sufficient row operations on the matrix to eliminate all zero entries in the first three columns.

Section 1.4

- 1.4.2 The product is not defined because the number of columns (1) in the \( 3 \times 1 \) matrix does not match the number of entries (2) in the vector.
- 1.4.4 \( Ax = \begin{bmatrix} 7 \\
8 \end{bmatrix} \)
- 1.4.14 No. The equation \( Ax = u \) has no solution.
- 1.4.23 a: False  b: True  c: False  d: True  e: True  f: True
- 1.4.24 a: True  b: True  c: True  d: True  e: False  f: True
- 1.4.26 \( 3u - 5v - w = 0 \) can be rewritten as \( 3u - 5v = w \), thus a solution is \( x_1 = 3, \, x_2 = -5 \).

Section 1.5

- 1.5.16 \( x = \begin{bmatrix} -5 \\
3 \\
0
\end{bmatrix} + x_3 \begin{bmatrix} -4 \\
3 \\
1
\end{bmatrix} = p + x_3q \). The solution set is the line through \( p \) parallel to \( q \).
- 1.5.23 a: True  b: False  c: False  d: False  e: False
- 1.5.24 a: False  b: True  c: True  d: True  e: False
- 1.5.30 a: Yes  b: No
- 1.5.32 a: Yes  b: Yes

Section 1.7

- 1.7.2 Linearly independent.
- 1.7.8 Linearly dependent.
- 1.7.21 a: False  b: False  c: True  d: True
- 1.7.22 a: True  b: False  c: True  d: False
Section 1.8

- 1.8.10 \( x_3 \begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \end{bmatrix} \)

- 1.8.12 No, because the system represented by \([A \mid b]\) is inconsistent.

- 1.8.20 \( \begin{bmatrix} -2 & 7 \\ 5 & -3 \end{bmatrix} \)

- 1.8.30 Let \( T(x) = Ax + b \) for \( x \in \mathbb{R}^n \). If \( b \) is not zero, \( T(0) = A0 + b = b \neq 0 \). (Actually one can show that \( T \) fails both properties of linear transformations)

Section 1.9

- 1.9.2 \( \begin{bmatrix} 1 & 4 & -5 \\ 3 & -7 & 4 \end{bmatrix} \)

- 1.9.4 \( \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \)

- 1.9.18 \( \begin{bmatrix} -3 & 2 \\ 1 & -4 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \)

- 1.9.20 \( \begin{bmatrix} 2 & 0 & 3 & -4 \end{bmatrix} \)

- 1.9.26 The transformation in Exercise 2 is not one-to-one, by Theorem 12, because the standard matrix is \( 2 \times 3 \), so it has linearly dependent columns. However, the matrix has a pivot in each row and so the columns span \( \mathbb{R}^2 \). By Theorem 12, the transformation maps \( \mathbb{R}^3 \) onto \( \mathbb{R}^2 \).

- 1.9.28 The standard matrix of the transformation \( T \) in Exercise 14 has linearly independent columns, because Figure 6 shows that \( a_1 \) and \( a_2 \) are not multiples. So \( T \) is one-to-one, by Theorem 12. Also, from Figure 6, \( a_1 \) and \( a_2 \), being linearly independent, span \( \mathbb{R}^2 \). Thus \( T \) maps \( \mathbb{R}^2 \) onto \( \mathbb{R}^2 \).

Section 2.1

- 2.1.4 \( A - 5I_3 = \begin{bmatrix} 4 & -1 & 3 \\ -8 & 2 & -6 \\ -4 & 1 & 3 \end{bmatrix} \) \( (5I_3)A = \begin{bmatrix} 45 & -5 & 15 \\ -40 & 35 & -30 \\ -20 & 5 & 40 \end{bmatrix} \)

- 2.1.20 The second column of \( AB \) is also all zeros because \( Ab_2 = A0 = 0 \).

- 2.1.22 If the columns of \( B \) are linearly dependent, then there exists a nonzero vector \( x \) such that \( Bx = 0 \). From this, \( A(Bx) = A0 \) and \( (AB)x = 0 \) by associativity. Since \( x \) is nonzero, the columns of \( AB \) must be linearly dependent.
• 2.1.24 Take any $b \in \mathbb{R}^m$. By hypothesis, $A(Db) = b$. Thus, the vector $x = Db$ satisfies $Ax = b$. This proves that the equation $Ax = b$ has a solution for each $b \in \mathbb{R}^m$. By Theorem 4 (Section 1.4), $A$ has a pivot position in each row. Since each pivot is in a different column, $A$ must have at least as many columns as rows.

Section 2.2

• 2.2.9 a: True b: False c: False d: True e: True

• 2.2.10 a: False b: True c: True d: True e: False

• 2.2.18 Left multiply each side of $A = PBP^{-1}$ by $P^{-1}$:

$$P^{-1}A = P^{-1}PBP^{-1} = IBP^{-1} = BP^{-1}.$$  

Then right multiply the result by $P$:

$$P^{-1}AP = BP^{-1}P = BI = B.$$  

• 2.2.22 Suppose $A$ is invertible. By Theorem 5, the equation $Ax = b$ has a solution (in fact, a unique one) for each $b$. By Theorem 4 in Section 1.4, the columns of $A$ span $\mathbb{R}^n$.

Section 2.3

• 2.3.11 a: True b: True c: False d: True e: True

• 2.3.12 a: True b: True c: True d: False e: True

• 2.3.14 If $A$ is lower triangular with nonzero entries on the diagonal, then these $n$ diagonal entries can be used as pivots to produce zeros below the diagonal. Thus $A$ has $n$ pivots and so is invertible, by the Invertible Matrix Theorem. If one of the diagonal entries in $A$ is zero, $A$ will have fewer than $n$ pivots and hence it will be singular.

Section 2.4

• 2.3.11 a: True b: False

Section 2.8

• 2.3.21 a: False b: True c: False d: True e: True

Section 5.1

• 5.1.2 Yes

• 5.1.4 Yes, $\lambda = 3 + \sqrt{2}$

• 5.1.6 Yes, $\lambda = -2$
• **5.1.8** Yes, \[
\begin{bmatrix}
3 \\
2 \\
1
\end{bmatrix}
\]

• **5.1.18** 4, 0, -3.

• **5.1.20** \( \lambda = 0 \). Eigenvectors for \( \lambda = 0 \) have entries that produce linear dependence relations between the columns of \( A \). Any nonzero vector (in \( \mathbb{R}^3 \)) whose entries sum to 0 will work. Find any two such vectors that are not multiples: for instance, take (1,1,-2) and (1,-1,0).

• **5.1.21**
  - a: False
  - b: True
  - c: True
  - d: True
  - e: False

**Section 5.2**

• **5.2.2** \( \lambda^2 - 10\lambda + 16 = 0, \lambda = 2, 8 \)

• **5.2.4** \( \lambda^2 - 8\lambda + 3 = 0, \lambda = 4 \pm \sqrt{13} \)

• **5.2.6** \( \lambda^2 - 11\lambda + 40 = 0, \) no real eigenvalues.

• **5.2.8** \( \lambda^2 - 10\lambda + 25 = 0, \lambda = 5 \)