TEST n.1

Instructions: Show all the work needed to obtain your answers.

Please note: you are expected to obey the honor code while taking this Test.

Exercise 1. (15 pts) Compute, when possible, the following matrix-vector products

a. \[
\begin{bmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 \\
2
\end{bmatrix}
\]

b. \[
\begin{bmatrix}
1 & 2 & 0 & 1 \\
3 & 1 & 2 & 0 \\
0 & 3 & 1 & 2 \\
1 & 0 & 3 & 1
\end{bmatrix}
\begin{bmatrix}
1 \\
0 \\
2 \\
1
\end{bmatrix}
\]

c. \[
\begin{bmatrix}
1 & 4 \\
2 & 3 \\
5 & 6
\end{bmatrix}
\begin{bmatrix}
2 \\
1
\end{bmatrix}
\]

2. Cannot be done

b. \[
\begin{bmatrix}
1+1 \\
3+4 \\
2+2 \\
1+6+4
\end{bmatrix}
= \begin{bmatrix}
2 \\
7 \\
4 \\
8
\end{bmatrix}
\]

C. \[
\begin{bmatrix}
2+4 \\
4+3 \\
10+6
\end{bmatrix}
= \begin{bmatrix}
6 \\
7 \\
16
\end{bmatrix}
\]
Exercise 2. (20 points) Let $A$ be a matrix with 3 rows and 6 columns, and let $b$ be a vector in $\mathbb{R}^3$. Assume that the reduced echelon form of the augmented matrix $[A|b]$ has the following pattern

\[
\begin{bmatrix}
1 & * & 0 & 0 & | & * \\
0 & 0 & 1 & 0 & | & * \\
0 & 0 & 0 & 0 & | & 1
\end{bmatrix}
\]

1. Does the linear system $Ax = b$ have solutions? If so, how many? Justify your answer.
2. Do the column of $A$ span the whole $\mathbb{R}^3$? Why yes or why not?
3. Does the homogeneous system $Ax = 0$ have nontrivial solutions? Why yes or why not?

1. Yes, the system features infinitely many solutions. It has solution because there is a pivot position in every row, and those are infinitely many because $x_2, x_3$, and $x_5$ are free variables.

2. Yes, there are three linearly independent columns $\{41, \, 44, \, 64\}$.

3. Yes, because there are free variables.
Exercise 3. (35 pts) Given

\[ A = \begin{bmatrix} 1 & 1 \\ h & 2 \end{bmatrix}, \quad b = \begin{bmatrix} k \\ 4 \end{bmatrix} \]

Determine the values of \( h \) and \( k \), if any, such that the linear system \( Ax = b \) is

1. inconsistent;
2. consistent with infinitely many solutions:
   (a) compute the general solution of the homogeneous system \( Ax = 0 \); give the parametric formulation of this solution and its geometrical interpretation;
   (b) find the solution of the system in terms of the sum of a particular solution and the general solution of the homogeneous system; give the geometrical interpretation of such solution;
3. consistent with a unique solution;
4. consistent with one solution such that \( x_1 = 0 \);

1. Row reduce system \([A|b]\):

\[
\begin{bmatrix}
1 & 1 & | & k \\
 h & 2 & | & 4
\end{bmatrix} \xrightarrow{R_2-hR_1} \begin{bmatrix}
1 & 1 & | & k \\
 0 & 2-h & | & 4-hk
\end{bmatrix}
\]

The system is inconsistent when \( 2-h=0 \) and \( 4-hk\neq0 \), i.e. for \( h=2 \) and \( k \neq 2 \).

2. The system is consistent with infinitely many solutions when \( 2-h=0 \) and \( 4-hk=0 \), i.e. \( h=2 \) and \( k = 2 \)

(a) \[ \begin{bmatrix} 1 & 1 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix} \Rightarrow x_1 + x_2 = 0 \Rightarrow \begin{cases} x_1 = -x_2 \\ x_2 \text{ free} \end{cases} \]

The parametric form is

\[
\overrightarrow{X}_{\text{hom}} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} t \quad t \in \mathbb{R}
\]

which represents a straight line through the origin, parallel to vector \([-1\]

\[x_2 \]

\[x_1 \]
(b) \[
\begin{bmatrix}
1 & 1 \\
0 & 0
\end{bmatrix}
\Rightarrow x_1 + x_2 = 2 \Rightarrow \begin{cases}
x_1 = 2 - x_2 \\
x_2 \text{ free}
\end{cases}
\]

The parametric form is:

\[
\vec{x} = \vec{x}_p + \vec{x}_\text{hom} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} t \quad t \in \mathbb{R}
\]

which represent a straight line through \( \begin{bmatrix} 2 \\ 0 \end{bmatrix} \) parallel to vector \( \begin{bmatrix} -1 \\ 1 \end{bmatrix} \).

3. The system is consistent with a unique solution whenever \( 2 - h \neq 0 \), i.e. \( h \neq 2 \).

4. We have:

\[
\begin{cases}
x_1 + x_2 = k \\
(2 - h)x_2 = 4 - hk
\end{cases}
\]

Assume \( 2 - h \neq 0 \), then, to have \( x_1 = 0 \) we must have:

\[
\begin{cases}
x_2 = k \\
x_2 = \frac{4 - hk}{2 - h}
\end{cases}
\]

To have consistency, it has to be \( k = \frac{4 - hk}{2 - h} \), i.e.

\[
2k - hk = 4 - hk \quad \Leftrightarrow \quad 2k = 4 \quad \Leftrightarrow \quad k = 2
\]

On the other hand, if \( 2 - h = 0 \), it must be \( k = 2 \) for the system to be consistent, and we have \( x_1 + x_2 = 2 \), which features the solution:

\[
\begin{cases}
x_1 = 0 \\
x_2 = 2
\end{cases}
\]
Exercise 4. (30 pts) Given the vectors

\[
\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix},
\]

determine whether vector \( \mathbf{b} \in \text{Span} \{ \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \} \) or not. If so, express \( \mathbf{b} \) as a linear combination of \( \{ \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \} \).

The vector \( \mathbf{b} \) is in \( \text{Span} \{ \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \} \) if and only if the system

\[
\mathbf{V} \mathbf{x} = \mathbf{b},
\]

where \( \mathbf{V} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix} \) is consistent.

\[
\begin{bmatrix} 1 & 2 & -1 & | & 5 \\ 2 & 0 & 3 & | & 1 \\ 1 & -1 & 1 & | & 0 \end{bmatrix} \xrightarrow{R_2 - 2R_1} \begin{bmatrix} 1 & 2 & -1 & | & 5 \\ 0 & -4 & 5 & | & -9 \\ 0 & -3 & 2 & | & -5 \end{bmatrix} \xrightarrow{4R_3 - 3R_2} \begin{bmatrix} 1 & 2 & -1 & | & 5 \\ 0 & 4 & -5 & | & 9 \\ 0 & 0 & -7 & | & 7 \end{bmatrix}
\]

we are thus reduced to system

\[
\begin{cases}
X_1 + 2X_2 - X_3 = 5 \\
4X_2 - 5X_3 = 9 \\
-7X_3 = 7
\end{cases} \quad \Rightarrow \quad \begin{cases}
X_3 = -1 \\
4X_2 - 5X_3 = 9 - 5 = 4 \Rightarrow X_2 = 1 \\
X_1 = 5 - 2X_2 + X_3 = 5 - 2 - 1 = 2
\end{cases}
\]

The system is consistent, thus \( \mathbf{b} \in \text{Span} \{ \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \} \), and we have

\[
\mathbf{b} = 2 \mathbf{v}_1 + \mathbf{v}_2 - \mathbf{v}_3
\]