Pebbling Graphs of Fixed Diameter

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Abstract

Given a configuration of indistinguishable pebbles on the vertices of a connected graph \( G \) on \( n \) vertices, a pebbling move is defined as the removal of two pebbles from some vertex, and the placement of one pebble on an adjacent vertex. The \( m \)-pebbling number of a graph \( G \), \( \pi_m(G) \), is the smallest integer \( k \) such that for each vertex \( v \) and each configuration of \( k \) pebbles on \( G \) there is a sequence of pebbling moves that places at least \( m \) pebbles on \( v \). When \( m = 1 \), it is simply called the pebbling number of a graph.

We prove that if \( G \) is a graph of diameter \( d \) and \( k,m \geq 1 \) are integers, then \( \pi_m(G) \leq f(k)n + 2k + d_m + (2k(2d - 1) - f(k))\dom_k(G) \), where \( \dom_k(G) \) denotes the size of the smallest distance \( k \) dominating set, that is the smallest subset of vertices such that every vertex is at most distance \( k \) from the set, and, \( f(k) = (2^k - 1)/k \). This generalizes the work of Chan and Godbole [4] who proved this formula for \( k = m = 1 \).

As a corollary, we prove that \( \pi_m(G) \leq f(\lceil d/2 \rceil)n + O(m + \sqrt{\log n}) \). Furthermore, we prove that if \( d \) is odd, then \( \pi_m(G) \leq f(\lceil d/2 \rceil)n + O(m) \), which in the case of \( m = 1 \) answers for odd \( d \), up to a constant additive factor, a question of Bukh [3] about the best possible bound on the pebbling number of a graph with respect to its diameter.

1 Introduction

A recent development in graph theory, suggested by Lagarias and Saks (via a private communication to Chung), is called pebbling. Pebbling was first introduced into the literature by Chung who computed the pebbling number of Cartesian products to give a combinatorial proof of the following number-theoretic statement of Kleitman and Lemke.

**Theorem 1.** [5][14] Let \( Z_n \) be the cyclic group on \( n \) elements and let \( |g| \) denote the order of a group element \( g \in Z_n \). For every sequence \( g_1, g_2, \ldots, g_n \) of (not necessarily distinct) elements of \( Z_n \), there exists a zero-sum subsequence \( (g_k)_{k \in K} \), such that \( \sum_{k \in K} \frac{|g_k|}{|g_k|} \leq 1 \).

Chung developed the pebbling game to give a more natural proof of this theorem. Theorems of this type are an important area of study in number theory as they generalize zero-sum theorems such as the Erdős–Ginzburg–Ziv [9] theorem. Geroldinger [10] and then Elledge and Hurlbert [8] generalized Theorem 1 to abelian groups. The latter work used graph pebbling to do so and also generalized the goal of zero-sum to a sum living in a given normal subgroup. Indeed, over the last twenty years, pebbling has developed into its own subfield [12, 13], with over sixty papers.

A pebbling configuration on a graph is a distribution of indistinguishable objects called pebbles on vertices of that graph. That is, a pebbling configuration \( p \) on a graph \( G \) is a function \( p : V(G) \to \mathbb{N} \cup \{0\} \), where \( p(v) \) is the number of pebbles on \( v \) in \( p \). A pebbling move is defined as the removal of two pebbles from some vertex and the subsequent placement

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of one pebble on an adjacent vertex. Hence, a pebbling move transforms one pebbling configuration to a different pebbling configuration.

We say the ordered pair \((G, r)\) is a rooted graph if \(G\) is a graph and \(r \in V(G)\). We say a pebbling configuration \(p\) is \(m\)-potent for a rooted graph \((G, r)\), if there exists a pebbling configuration \(p'\) obtained by a sequence of pebbling moves from \(p\) such that \(r\) has at least \(m\) pebbles in \(p'\). We say a pebbling configuration \(p\) is \(m\)-impotent if there does not exist such a pebbling configuration. As in [5], we define the \(m\)-pebbling number, \(\pi_m(G)\), to be the least integer \(k\) such that, for any choice of root \(r \in V(G)\) and any initial configuration \(p\) of \(k\) pebbles, \(p\) is \(m\)-potent for \((G, r)\). The pebbling number refers to the 1-pebbling number of a graph. Notice that a trivial lower bound for \(\pi_1(G)\) is \(|V(G)|\): Choose \(r \in V(G)\) and let \(p(r) = 0\) and \(p(v) = 1\) for all \(v \neq r\). Then \(p\) is 1-impotent for \((G, r)\).

The diameter of a graph can also yield lower bounds on its pebbling number. For instance, because the pebbling number of a path on \(d\) vertices is \(2^{d-1}\), then the pebbling number of graphs with diameter \(d\) must be at least \(2^d\). It is then a natural question to ask whether restricting the diameter can give an upper bound on the pebbling number. To that end, define \(\pi_m(n, d)\) to be the maximum \(m\)-pebbling number of a diameter \(d\) graph on \(n\) vertices.

For graphs of diameter two much is known. Pachter, Snevily and Voxman in [15] proved that \(\pi_1(n, 2) = n + 1\). Clarke, Hochberg and Hurlbert in [6] classified graphs of diameter two whose pebbling number is \(n + 1\). Curtis, et al. [7] proved that \(\pi_m(n, 2) \leq n + 7m - 6\) and conjectured that \(\pi_m(n, 2) \leq n + 4m - 3\), which was recently proved by Herscovici, et al. [11].

As for graphs of larger diameter, more recent results have provided insight for graphs of diameter three and four. Bukh [3] proved that \(\pi_1(n, 3) = 3n/2 + O(1)\). Postle, Streib and Yerger [17] proved an exact bound for \(\pi_1(n, 3)\), namely that \(\pi_1(n, 3) = \lfloor 3n/2 \rfloor + 2\). They also gave shorter proofs of the aforementioned diameter two results using their new techniques. Furthermore, they proved that \(\pi_1(n, 4) = 3n/2 + O(1)\). As for general diameter results, Bukh proved that \(\pi_1(n, d) \leq 2^{\left\lfloor d/2 \right\rfloor - 1}n + O(\sqrt{n})\). The best known lower bound is \(\pi_m(n, d) \geq f(\lfloor d/2 \rfloor)(n - (d + 1)) + 2^m, which comes from a generalization of the example in [3].

In section two, we define branches, which were previously introduced in [17], and prove several fundamental results about them. In section three, we use these results to bound pebbling numbers in terms of domination numbers.

Let \(G\) be a graph. We say that \(S\) is a \(k\)-dominating set if every vertex in \(G\) is either in \(S\) or is at most distance \(k\) from some vertex in \(S\). The \(k\)-domination number, denoted by \(dom_k(G)\), is the size of the smallest \(k\)-dominating set in \(G\). When \(k = 1\), this is the domination number.

We prove the following theorem relating pebbling number and \(k\)-domination numbers.

**Theorem 2.** Let \(G\) be a graph on \(n\) vertices of diameter \(d\). For all \(k, m \geq 1\), \(\pi_m(G) \leq f(k)n + 2^{k+4}(m - 1) + (2^{k}(2^d - 1) - f(k))dom_k(G)\), where \(f(k) = \frac{(2^k - 1)}{k}\).

This generalizes a result of Chan and Godbole [4] who proved that \(\pi_1(G) \leq n + (2^{d+1} - 3)dom(G)\).

As a corollary to this theorem, we obtain a bound on \(\pi_m(n, d)\).

**Corollary 1.** If \(G\) is a graph on \(n\) vertices of diameter \(d\), then \(\pi_m(G) \leq f(\lfloor d/2 \rfloor)n + 2^{\lfloor d/2 \rfloor + [d/2]}(m - 1) + (2^{[d/2]}(2^d - 1) - f(\lfloor d/2 \rfloor))\sqrt{n \ln n}.

Hence, \(\pi_m(n, d) \leq f(d/2)n + O(m + \sqrt{n \ln n})\), which is best possible up to a sublinear asymptotic factor. It is worth comparing this to the recent work of Herscovici, et al. who proved that \(\pi_m(n, d) \leq f(d)(n - 1) + 2^d(m - 1) + 1\).

We also prove the following theorem:

**Theorem 3.** If \(d\) is an odd positive integer, then \(\pi_m(n, d) \leq f(\lfloor d/2 \rfloor)n + O(m)\).
Given the lower bounds mentioned above this proves that

**Corollary 2.** If \( d \) is an odd positive integer, then \( \pi_1(n,d) = \theta(f([d/2])n) \).

## 2 Branches

Let \( S \) be a subset of the vertices. We say that a spanning forest \( T \) of \( G \) is a breadth-first search (BFS) spanning forest of \( G \) with root set \( S \) if, for every vertex \( v \in V(G) \), the shortest path from \( v \) to \( S \) in \( T \) is also a shortest path from \( v \) to \( S \) in \( G \), and every vertex \( v \) in \( S \) is contained in a different component of \( T \). We will use the standard notions from BFS trees of descendant, child, parent, and ancestor for BFS forests as well. We also let \( d(u,v) \) denote the distance between two vertices.

**Definition.** We say the ordered triple \((B, p, w)\) is a branch if \((B, w)\) is a rooted tree and \( p \) is a pebbling configuration on \( V(B) \). Where \( p \) and \( w \) are understood, we will say that \( B \) is branch. The depth of a branch \((B, p, w)\), to be denoted by \( d(B) \), is the maximum distance in \( B \) from a vertex in \( B \) to \( w \).

We also define \( p(B) \) to be the number of pebbles in the branch, that is, \( \sum_{v \in V(B)} p(v) \).

**Definition.** Let \((B, p, w)\) be a branch of depth \( k > 0 \). We define the truncation of \( B \) to be the branch \((B', p', w)\) obtained from \( B \) by making pebbling moves to move as many pebbles as possible from all the vertices of depth \( k \) to their parents and then deleting the vertices of depth \( k \). If \( i \leq k \), we define the \( i \)-truncation of \( B \), to be denoted by \( B^{(i)} \), as the branch obtained from \( B \) by successively truncating it \( i \) times.

**Definition.** We define the potency of branch \((B, p, w)\), to be denoted as \( p(B) \), as \( p(B^{d(B)}) \). Similarly we define the capacity, denoted by \( c(B) \), as \( \lfloor p(B) / 2 \rfloor \).

If \( u \) is a vertex in a branch \( B \) we will let \( B[u] \) denote the subbranch \((B_u, p, u)\) of \( B \), where \( B_u \) is the subtree of \( B \) induced by \( u \) and all of its descendants. A branch \((B, p, w)\) is irreducible if for all vertices \( v \in B \), where \( v \neq w \), \( B[v] \) has nonzero capacity. If \( B \) is not irreducible, we will say that \( B \) is reducible. If \( B \) is reducible and \( u \in V(B) \) such that \( u \neq w \) and \( B[u] \) has zero capacity, then \( B \) may be decomposed into two branches \( B \setminus B[u] \) and \( B[u] \). Continuing this process, we may decompose \( B \) into irreducible branches; moreover this decomposition is unique as the roots of these branches must exactly correspond to the vertices \( v \in V(B) \) such that \( u = w \) or \( B[u] \) has zero capacity. We refer to this decomposition as the irreducible decomposition of \( B \).

**Definition.** We define the function \( F(k) \) to be the supremum of \( \frac{p(B)}{|V(B)|} \) over all branches \((B, p, w)\) of depth at most \( k - 1 \). For all \( k \geq 1 \), we define the \( k \)-excess of a branch \( B \), denoted \( X_k(B) \), to be \( p(B) - F(k)|V(B)| \).

**Proposition 1.** There are only finitely many irreducible branches of depth at most \( d \) and potency \( l \).

**Proof.** We proceed by induction on \( d \). If \( d = 0 \), then such a branch is simply a vertex and the number of pebbles on that vertex is \( l \). So we may assume that \( d \geq 1 \). Now in an irreducible branch of potency \( l \), the root \( w_0 \) has at most \( l \) children. The subbranches induced by the children of \( w_0 \) have depth at most \( d - 1 \). As the number of possible such subbranches is finite by induction and the range of possible values for \( p(w_0) \) is also finite, there are at most a finite number of possible branches of depth at most \( d \) and potency \( l \).

**Lemma 1.** \( F(k) \) is equal to the maximum of \( \frac{p(B)}{|V(B)|} \) over all irreducible branches of zero capacity and depth at most \( k - 1 \).
Proof. By Proposition 1, the supremum over irreducible branches is indeed a maximum. Let \((B,p,w)\) be a branch of zero capacity and depth at most \(k-1\). Consider the irreducible decomposition of \(B\) into irreducible branches \(B_1, \ldots, B_t\), which have zero capacity and depth at most \(k-1\). As \(p(B) = \sum_{i=1}^{t} p(B_i)\), \(\frac{p(B)}{|V(B)|} = \sum_{i=1}^{t} \frac{p(B_i)}{|V(B)|}\). If we let \(c\) denote the maximum of \(\frac{p(B_i)}{|V(B)|}\) over irreducible branches of zero capacity and depth at most \(k-1\), then this is at most \(c \sum_{i=1}^{t} \frac{|V(B_i)|}{|V(B)|}\). Since \(|V(B)| = \sum_{i=1}^{t} |V(B_i)|\), this is at most \(c\) as desired. \(\square\)

Lemma 2. For all \(d \leq k\), the supremum of \(X_k(B)\) over all branches of depth at most \(d\) and potency \(l\) is equal to the maximum of \(X_k(B)\) over all irreducible branches of depth at most \(d\) and potency \(l\).

Proof. Let \(c\) be the maximum of \(X_k(B)\) over all irreducible branches of depth at most \(d\) and potency \(l\). Such a maximum exists by Proposition 1. It suffices to prove that if \((B,p,w)\) is a branch of depth at most \(d\) and potency \(l\), then \(X_k(B) \leq c\). We proceed by induction on \(d\) and then induction on \(|V(B)|\). If \(B\) is irreducible, this follows from the definition of \(c\). If \(d = 0\), then \(B\) is simply a vertex and so irreducible and the lemma follows.

So we may assume that \(d \geq 1\) and \(B\) is not irreducible. Then there exists \(u \in V(B)\setminus \{w\}\) such that \(B[u]\) has capacity zero. Yet, \(B' = B \setminus B[u]\) is a branch with depth at most \(d\), potency \(l\), and a smaller number of vertices. In addition, \(X_k(B') = X_k(B) - X_k(B[u])\). Since \(B[u]\) has depth at most \(d - 1\) which is at most \(k - 1\), \(X_k(B[u]) \leq 0\) by the definition of \(F(k)\). Hence, \(X_k(B) \leq X_k(B') \leq c\) as desired. \(\square\)

Lemma 3. Let \(d, l\) be non-negative integers and let \(k\) be an integer such that \(k \geq d\). Suppose that \((B,p,w_0)\) is an irreducible branch of depth at most \(d\) and potency \(l\) such that \(B\) has maximum \(k\)-excess and, subject to that condition, has a minimum number of vertices. It follows that \(B = w_0w_1 \ldots w_{\ell(B)}\) is a path such that \(p(w_i) = 0\) for all \(i, 0 \leq i < \ell(B)\).

Proof. We claim that if \(u \in V(B)\) is not a leaf, then \(p(u) = 0\). Suppose not. Define a new pebbling configuration \(p'\) on \(V(B)\) as follows. Let \(v\) be a child of \(u\). Let \(p'(u) = p(u) - 1\), \(p'(v) = p(v) + 2\) and \(p'(z) = p(z)\) for all other vertices \(z \neq u, v\). The branch \((B,p',w_0)\) has depth at most \(d\) and potency \(l\). However, \(p'(B) > p(B)\) and thus \((B,p',w_0)\) has a larger \(k\)-excess, a contradiction.

Finally we claim that all vertices in \(B\) have at most one child. Suppose not. Let \(v\) be a vertex with at least two children but such that every descendant of \(v\) has at most one child. Let \(u_1, u_2\) be two children of \(v\). Let \(t_1\) be the descendant of \(u_1\) that is a leaf and \(t_2\) be the descendant of \(u_2\) that is a leaf. We may assume without loss of generality that \(d(v,t_1) \geq d(v,t_2)\). Define a new pebbling configuration \(p'\) on \(V(B)\) as follows. Let \(q\) be the largest integer such that \(p(t_2) \geq 2q^{d(v,t_2)}\). Let \(p'(t_2) = p(t_2) - q2^{d(v,t_2)}\), \(p'(t_1) = p(t_1) + q2^{d(v,t_1)}\) and \(p'(z) = p(z)\) for all other vertices \(z \neq t_1, t_2\). The branch \((B,p',w_0)\) has depth at most \(d\) and potency \(l\). Moreover, \(p'(B) \geq p(B)\) and thus must also have maximum excess. However, the induced subbranch \(B[u_2]\) of \((B,p',w_0)\) has capacity zero since \(p'(t_2) \leq 2q^{d(v,t_2)} - 1\) and \(p'(x) = 0\) for all other \(x \in B[u_2]\). Hence \((B,p',w_0)\) is not irreducible, a contradiction. \(\square\)

Corollary 3. For all \(k \geq 1\), \(F(k) = f(k)\).

Proof. By Lemma 1, the maximum \(k\)-excess among branches of depth at most \(k-1\) and zero capacity is attained at some irreducible branch \((B,p,w)\). By Lemma 3, we may assume that \(B\) is a path \(w_0w_1 \ldots w_{\ell(B)}\) and \(p(w_i) = 0\) for all \(i, 0 \leq i < \ell(B)\). Now \(p\) is 2-impotent for \((B,w_0)\) if and only if \(p(w_{\ell(B)}) \leq 2^{d(B)+1} - 1\). The maximum \(k\)-excesses given depth \(d(B)\) would thus be obtained when \(p(w_{k-1}) = 2^{d(B)} - 1 \text{ and } X_k(B) = p(B) - F(k)|V(B)| = 2^{d(B)} - 1 - F(k)(d(B) + 1)\). However as \(B\) has maximum \(k\)-excess, \(X_k(B) = 0\). Thus, \(F(k) = \frac{2^{d(B)+1} - 1}{d(B)+1}\). Certainly this is maximized when \(d(B)\) is maximized, that is when \(d(B) = k - 1\). Thus \(F(k) = \frac{2^{k-1}}{k} = f(k)\) as desired. \(\square\)
Corollary 5. The maximum $k$-excess over branches of depth at most $k$ and potency $l$ is $2^k l - f(k)$. Hence, if $B$ is branch of depth at most $k$, then $X_k(B) \leq 2^k \overline{p}(B) - f(k)$.

Proof. By Lemma 1, the maximum $k$-excess among branches of depth at most $k - 1$ and potency $l$ is attained at some irreducible branch $(B, p, w_0)$. By Lemma 3, we may assume that $B$ is a path $w_0 w_1 \ldots w_{d(B)}$ and $p(w_i) = 0$ for all $i$, $0 \leq i < d(B)$. As $B$ has potency $l$, $p(w_{d(B)}) = 2^d(B)(l+1)-1$. The maximum $k$-excess for a branch of depth $d(B)$ would thus be obtained when $p(w_{d(B)}) = 2^d(B)(l+1)-1$. Hence, $X_k(B) = p(B) - f(k)(|V(B)| = 2^d(B)(l+1)-1 - (\frac{2^k-1}{k})(d(B)+1)$. It is not hard to see that the maximum $k$-excess is obtained when the depth is maximized, that is when $d(B) = k-1$ and hence $X_k(B) = 2^k(l+1) - F(k)k$.

Hence, $X_k(B) = 2^k(l+1) - 1 - (\frac{2^k-1}{k})(k+1) = 2^k l - f(k)$ as desired.

If $B$ is a branch of depth at most $k$, its potency is $\overline{p}(B)$. Hence, $X_kB$ is the most the maximum $k$-excess over branches of depth at most $k$ and potency $\overline{p}(B)$ which is $2^k \overline{p}(B) - f(k)$. \(\square\)

3 Dominating Sets

The following theorem was proved by Arnautov and independently by Payan.

Theorem 4. [2][16] If $G$ is a graph with minimum degree $\delta(G)$, then $dom(G) \leq n(1 + \ln(\delta(G) + 1))/\delta(G) + 1)$.

The following are two useful corollaries of this theorem. The first was proved asymptotically by Al-Yakoob and Tuza [1] with a slightly better bound.

Corollary 5. If $G$ is a graph of diameter at most two on $n \geq 3$ vertices, then $dom(G) \leq \sqrt{n \ln n}$.

Proof. Let $v \in V(G)$. Notice that $N(v)$ is a dominating set in $G$ as $G$ is diameter two. Hence, $dom(G) \leq \delta(G)$. Thus if $\delta(G) \leq \sqrt{n \ln n}$, Corollary 5 holds. So suppose $\delta(G) > \sqrt{n \ln n}$. By Theorem 4, $dom(G) \leq n(1 + \ln(\delta(G) + 1))/\delta(G) + 1) \leq n(1 + \ln(n)/2 + \ln \ln(n)/2)/\sqrt{n \ln n}$. However this is at most $n \ln(n)/\sqrt{n \ln n} = \sqrt{n \ln n}$ since $n \geq 3$, and Corollary 5 holds. \(\square\)

Corollary 6. If $G$ is a graph of diameter $d$ on $n \geq 3$ vertices, then $dom_{\lfloor d/2 \rfloor}(G) \leq \sqrt{n \ln n}$.

Proof. Apply Corollary 5 to the graph $G'$ where $V(G') = V(G)$ and there is an edge between two vertices $x$ and $y$ if and only if $d(x, y) \leq \lfloor d/2 \rfloor$. \(\square\)

Now we are prepared to prove the main theorem.

Proof of Theorem 2. Let $r \in V(G)$ and let $p$ be a pebbling configuration that is $m$-impotent for $(G, r)$. Let $S$ be a smallest $k$-dominating set in $G$. Let $T$ be a BFS spanning forest with root set $S \cup r$. For every $s \in S \cup r$, let $C_s$ denote the component of $T$ containing $s$. Notice that $(C_s, p, s)$ is a branch of depth at most $k$.

Note that $\sum_{v \in V(G)} p(v) = \sum_{s \in S \cup r} p(C_s) = f(k)n + \sum_{s \in S \cup r} X_k(C_s)$. Moreover, if $\overline{p}(C_s) \geq q2^d$ then $C_s$ can send $q$ pebbles to $r$ using only the pebbles in $C_s$. For all $s \in S$, let $q_s$ be the largest integer such that $\overline{p}(C_s) \geq q_s 2^d$. Let $q_r = \overline{p}(C_r)$. Since $p$ is $m$-impotent, it follows that $\sum_{s \in S} q_s + q_r \leq m - 1$. Thus, $\sum_{s \in S \cup r} \overline{p}(C_s) \leq \sum_{s \in S} (2^d q_s + 2^d - 1) + q_r \leq 2^d(m - 1) + (2^d - 1)(dom_k(G) - q_s)$. By Corollary 4, $X_k(C_s) \leq 2^d \overline{p}(C_s) - f(k)$ for all $s \in S \cup r$. Hence, $\sum_{v \in V(G)} p(v) \leq f(k)n + 2^d + (m - 1) + (2^d - 1) - f(k) = \overline{p}(C_r) - 2^d q_r - f(k)$.

As $f(k) \geq 1$ since $k \geq 1$ and $q_r \geq 0$, this at most one less than the formula desired in Theorem 2. Since $\pi_m(G) - 1$ is equal to the maximum number of pebbles over all $m$-impotent configurations, Theorem 2 holds. \(\square\)
Proof of Corollary 1. Apply Theorem 2 with \( k = \lceil d/2 \rceil \). By Corollary 6, \( \text{dom}_{\lfloor d/2 \rfloor}(G) \leq \sqrt{n \ln n} \). \( \square \)

Finally, we improve on our bound for odd \( d \) to obtain a bound that is best possible up to a constant additive factor.

**Theorem 5.** If \( G \) is a graph on \( n \) vertices of odd diameter \( d \), then \( \pi_m(G) \leq f([d/2])n + 2^{d/2}[d/2](m - 1) + 8^{d+4}/3 \).  

**Proof.** If \( d = 1 \), then \( \text{dom}(G) = 1 \) and Theorem 5 follows by Theorem 2 with \( k = 1 \). So suppose \( d \geq 3 \). Consider \( G^{[d/2]} \), the graph with vertex set \( V(G) \) where for all \( x, y \in V' \), \( x \) is adjacent to \( y \) if and only if \( d(x, y) \leq [d/2] \). Let \( \alpha = \delta(G^{[d/2]}) + 1 \). Note that the \( [d/2] \)-neighborhood of any vertex is a \( [d/2] \)-dominating set in \( G \). In other words, \( \text{dom}_{[d/2]}(G) \leq \alpha - 1 \). Yet we also know that any dominating set in \( G^{[d/2]} \) is a \( [d/2] \)-dominating set in \( G \). Hence by Theorem 4, \( \text{dom}_{\lfloor d/2 \rfloor}(G) \leq n \ln(\alpha)/\alpha \).

Now we condition on \( \alpha \). If \( \alpha \leq 16(2^{d/2}) \), apply Theorem 2 with \( k = \lceil d/2 \rceil \). We obtain the following bound as desired: \( \pi_m(G) \leq f([d/2])n + 2^{d/2}[d/2](m - 1) + 8^{d+4}/3 \).

If \( \alpha \geq 16(2^{d/2}) \), apply Theorem 2 with \( k = \lceil d/2 \rceil \). We obtain the following bound: \( \pi_m(G) \leq f([d/2])n + 2^{d/2}[d/2](m - 1) + n \ln(\alpha)/\alpha \). Under these assumptions, \( 2^{d/2}[d/2] \ln(\alpha)/\alpha \leq (\ln(16) + 3 \ln(2)d/2)/16 = \ln(2)(8 + 3d)/32 \). Note that for all \( d \geq 3 \) the difference between \( f([d/2]) \) and \( f([d/2]) \) is at least \( d/6 \). Since \( d/6 > \ln(2)(8 + 3d)/32 \) for all \( d \geq 3 \), we can merge the two linear terms into one to obtain the following bound: \( \pi_m(G) \leq f([d/2]) + 2^{d/2}[d/2](m - 1) \). \( \square \)

**References**


