Defn of a “nice” Transformation for 2-dim spaces $T$.

$T$ is given by $x = g(u,v)$, $y = h(u,v)$.

$T$ maps a set of points $S$ contained in the $uv$-plane onto a set of points $R$ contained in the $xy$-plane. $T(u,v) = (g(u,v), h(u,v))$ is a point in $R$ contained in the $xy$-plane; $T(u,v)$ is called the image of the point $(u,v)$ under $T$ in $R$.

Small Print:
“nice” means NO two points of $S$ have the same image under $T$ in $R$ AND Every point of $R$ is the image under $T$ of some point in $S$. 
Definition of a "nice" Transformation for 3-dim space $T$

$T$ is given by

$x = g(u,v,w)$
$y = h(u,v,w)$
$z = k(u,v,w)$

$T$ maps a set of points $B$ contained in the $uvw$-space onto a set of points $E$ contained in the $xyz$-space.

$T(u,v,w) = (g(u,v,w), h(u,v,w), k(u,v,w))$ is a point in $E$.

$T(u,v,w)$ is called the image of the point $(u,v,w)$ under $T$.

Small Point:

"nice" means no two points of $B$ have the same image under $T$ in $E$ and every point of $E$ is the image under $T$ of some point in $B$. 
Defn. The Jacobian of the transformation $T$.

2-dim: $T$ is given by $x = g(u, v)$ and $y = h(u, v)$.

\[
\frac{\partial (x, y)}{\partial (u, v)} = \det \begin{bmatrix}
\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\
\frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}
\end{bmatrix} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u}
\]

3-dim: $T$ is given by $x = g(u, v, w)$, $y = h(u, v, w)$, and $z = k(u, v, w)$.

\[
\frac{\partial (x, y, z)}{\partial (u, v, w)} = \det \begin{bmatrix}
\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\
\frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\
\frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w}
\end{bmatrix}
\]

\[
= \frac{\partial x}{\partial u} \left( \det \begin{bmatrix} \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\
\frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{bmatrix} \right) - \frac{\partial x}{\partial v} \left( \det \begin{bmatrix} \frac{\partial y}{\partial u} & \frac{\partial y}{\partial w} \\
\frac{\partial z}{\partial u} & \frac{\partial z}{\partial w} \end{bmatrix} \right) + \frac{\partial x}{\partial w} \left( \det \begin{bmatrix} \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\
\frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{bmatrix} \right)
\]
Thm. Change of Variables for integrals

Let $T$ be a "nice" transformation.
Let Jacobian of $T$ be nonzero. Let $f$ be continuous.

2-dim: $T$ maps $S$ onto $R,$

$$\iint_{S} f(T(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \, du \, dv = \iint_{R} f(x,y) \, dx \, dy.$$ 

3-dim: $T$ maps $B$ onto $E,$

$$\iiint_{B} f(T(u,v,w)) \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| \, du \, dv \, dw = \iiint_{E} f(x,y,z) \, dx \, dy \, dz.$$
Examples of transformations

2-dim: \( T_{\text{Polar}} \) is given by \( x = r \cos \theta \)
\[ y = r \sin \theta \]

The absolute value of the Jacobian of \( T_{\text{Polar}} \) is
\[
\left| \frac{\partial (x,y)}{\partial (r,\theta)} \right| = r
\]

3-dim: \( T_{\text{Cyl}} \) is given by \( x = r \cos \theta \)
\[ y = r \sin \theta \]
\[ z = z \]

The absolute value of the Jacobian of \( T_{\text{Cyl}} \) is
\[
\left| \frac{\partial (x,y,z)}{\partial (r,\theta,z)} \right| = r
\]

3-dim: \( T_{\text{Sph}} \) is given by \( x = r \sin \phi \cos \theta \)
\[ y = r \sin \phi \sin \theta \]
\[ z = r \cos \phi \]

The absolute value of the Jacobian of \( T_{\text{Sph}} \) is
\[
\left| \frac{\partial (x,y,z)}{\partial (r,\theta,\phi)} \right| = r^2 \sin \phi
\]