Defn: let \( y_1 \) and \( y_2 \) be functions.

The Wronskian of \( y_1 \) and \( y_2 \) is notated & defined below

\[
W(y_1, y_2) = (y_1)[y_2'] - (y_2)[y_1'].
\]

Note: One can see a determinant of a 2x2 matrix:

\[
W(y_1, y_2) = \det \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} = (y_1)[y_2'] - (y_2)[y_1']
\]

Thm:

Let \( y_1 \) and \( y_2 \) be solutions to the D.E.

\[
y'' + (b(t)) y' + (c(t)) y = 0
\]

Then \( W(y_1, y_2) \) is not zero at a to [maybe more than one t-value] if and only if

\[
y_{gen} = k_1 y_1 + k_2 y_2 \quad \text{is every solution to the D.E.}
\]

Note: here \( k_1 \) and \( k_2 \) are (real) numbers.
Thm Existence & Uniqueness of 2nd order IVPs

Consider the IVP

\[ \begin{cases} 
    y'' + (b(t)) y' + (c(t)) y = g(t) \\
    y(t_0) = \alpha_0 \\
    y'(t_0) = \beta_0 
\end{cases} \]

where \( b(t) \), \( c(t) \), and \( g(t) \) are continuous on an open interval \( I \) that contains \( t_0 \).

Then there is exactly one solution \( y = \varphi(t) \) of the IVP, and the solution exists throughout the interval \( I \).
Example

Let \( r_1 \) be different than \( r_2 \) (both real numbers).

\[
W(e^{r_1t}, e^{r_2t}) = (e^{r_1t})[r_2e^{r_2t}] - (e^{r_2t})[r_1e^{r_1t}]
\]
\[
= r_2e^{(r_1+r_2)t} - r_1e^{(r_1+r_2)t}
\]
\[
= (r_2 - r_1)e^{(r_1+r_2)t}.
\]

Note: since \( r_1 \neq r_2 \), \( r_2 - r_1 \) is not zero. Moreover, \( e^{(r_1+r_2)t} \neq 0 \)

Therefore \( W(e^{r_1t}, e^{r_2t}) \neq 0 \) for all \( t \)-values

What D.E. is \( y_1 = e^{r_1t} \) and \( y_2 = e^{r_2t} \) both solutions to?

\[
y'' + (-r_1-r_2)y' + (r_1r_2)y = 0
\]
Example

Let \( r \) be a real number.

\[
W(e^{rt}, te^{rt}) = \left( e^{rt} \right)[e^{rt} + r e^{rt}] - \left( te^{rt} \right)[r e^{rt}]
\]

\[
= e^{2rt} + rte^{rt} - rte^{2rt}
\]

\[
= e^{2rt}.
\]

Note: \( e^{2rt} \neq 0 \).

Therefore \( W(e^{rt}, te^{rt}) \neq 0 \) for all \( t \)-values.

What D.E. is \( y_1 = e^{rt} \) and \( y_2 = te^{rt} \) both solutions to?

\[
y'' + (-2r) y' + (r^2) y = 0
\]
Example

Let \( \lambda \) be a real number, and
let \( \mu \) be a non-zero real number.

\[
W(e^{\lambda t} \sin(\mu t), e^{\lambda t} \cos(\mu t)) = (e^{\lambda t} \sin(\mu t))[(\lambda e^{\lambda t} \cos(\mu t)) - (\mu e^{\lambda t} \sin(\mu t))] - (e^{\lambda t} \cos(\mu t))[(\lambda e^{\lambda t} \sin(\mu t)) + (\mu e^{\lambda t} \cos(\mu t))]
\]

\[= -\mu e^{2\lambda t}.
\]

Note: Since \( \mu \) is non-zero, \(-\mu e^{2\lambda t} \neq 0\).

Therefore \( W(e^{\lambda t} \sin(\mu t), e^{\lambda t} \cos(\mu t)) \neq 0 \) for all \( t \)-values.

What D.E. is \( y_1 = e^{\lambda t} \sin(\mu t) \) and \( y_2 = e^{\lambda t} \cos(\mu t) \) both solutions to?

\[
y'' + (-2\lambda) y' + (\lambda^2 + \mu^2) y = 0
\]