Combinatorial Problems

Today we look at a few simple Putnam problems involving combinatorial reasoning (counting, existence, and games) from 2005 to 2008.

We'll spend half our time looking at the problems; you should try to find an idea or two. We'll spend our remaining time discussing hints and solutions. For official solutions, read the third page. As usual, these problems and solutions are from Kedlaya's archive.

I. PROBLEMS

Look over the problems below. Try to identify one or more where you have some idea how to proceed. For a real Putnam session, I recommend you spend at least half an hour just on this!

2005 A–1: Show that every positive integer is a sum of one or more numbers of the form 2^r3^s, where r and s are nonnegative integers and no summand divides another. (For example, 23 = 9 + 8 + 6.)

2005 B–4: For positive integers m and n, let $f(m,n)$ denote the number of n-tuples $(x_1,x_2,\ldots,x_n)$ of integers such that $|x_1|+|x_2|+\cdots+|x_n|\leq m$. Show that $f(m,n)=f(n,m)$.

2006 A–2: Alice and Bob play a game in which they take turns removing stones from a heap that initially has n stones. The number of stones removed at each turn must be one less than a prime number. The winner is the player who takes the last stone. Alice plays first. Prove that there are infinitely many n such that Bob has a winning strategy. (For example, if $n=17$, then Alice might take 6 leaving 11; then Bob might take 1 leaving 10; then Alice can take the remaining stones to win.)

2007 A–3: Let k be a positive integer. Suppose that the integers $1,2,3,\ldots,3k+1$ are written down in random order. What is the probability that at no time during this process, the sum of the integers that have been written up to that time is a positive integer divisible by 3? Your answer should be in closed form, but may include factorials.

2008 A–2: Alan and Barbara play a game in which they take turns filling entries of an initially empty 2008×2008 array. Alan plays first. At each turn, a player chooses a real number and places it in a vacant entry. The game ends when all the entries are filled. Alan wins if the determinant of the resulting matrix is nonzero; Barbara wins if it is zero. Which player has a winning strategy?

II. HINTS

You won't get hints on a real exam, but these ideas that may help you with similar problems. Before looking at the solutions, see if you can make any further progress with these hints.

2005 A–1: Induction. (The representation does not have to be unique.)

2005 B–4: Find either an explicit formula or a symmetric recurrence for f.

2006 A–2: Otherwise, every positive integer is "close" to a prime.

2007 A–3: First leave out the multiples of three.

2008 A–2: State a simple winning strategy. It helps that n is even.

The next page has solutions, don't continue until you are ready to see them!

III. SOLUTIONS

These are solutions collected at Kedlaya's archive. Of course you don't have to find the same solutions, and you don't have to write in such a polished style. But if you did write out solutions, you might compare yours to these. In particular, did you define things clearly? Did you leave out some important step, or handle it incorrectly?

2005 A–1: We proceed by induction, with base case $1 = 2^0 3^0$. Suppose all integers less than $n-1$ can be represented. If n is even, then we can take a representation of $n/2$ and multiply each term by 2 to obtain a representation of n. If n is odd, put $m = \lfloor \log_3 n \rfloor$, so that $3^m \leq n < 3^{m+1}$. If $3^m = n$, we are done. Otherwise, choose a representation $(n-3^m)/2 = s_1 + \cdots + s_k$ in the desired form. Then $n = 3^m + 2s_1 + \cdots + 2s_k$, and clearly none of the $2s_i$ divide each other or 3^m. Moreover, since $2s_i \leq n - 3^m < 3^{m+1} - 3^m$, we have $s_i < 3^m$, so 3^m cannot divide $2s_i$ either. Thus n has a representation of the desired form in all cases, completing the induction.

Remark: This problem is due to Paul Erdős. The representation is not always unique: $11 = 2+9 = 3+8$.

2005 B–4: Define $f(m,n,k)$ as the number of n-tuples $(x_1, x_2, \ldots, x_n)$ of integers such that $|x_1| + \cdots + |x_n| \leq m$ and exactly k of $x_1, \ldots, x_n$ are nonzero. To choose such a tuple, we may choose the k nonzero positions, the signs of those k numbers, and then an ordered k-tuple of positive integers with sum $\leq m$. There are $\binom{n}{k}$ options for the first choice, and 2^k for the second. As for the third, we have $\binom{m}{k}$ options by a "stars and bars" argument: depict the k-tuple by drawing a number of stars for each term, separated by bars, and adding stars at the end to get a total of m stars. Then each tuple corresponds to placing k bars, each in a different position behind one of the m fixed stars.

We conclude that $f(m,n,k) = 2^k \binom{m}{k}\binom{n}{k} = f(n,m,k)$. Summing over k gives $f(m,n) = f(n,m)$.

2006 A–2: Suppose on the contrary that the set B of values of n for which Bob has a winning strategy is finite; for convenience, we include $n = 0$ in B, and write $B = \{b_1, \ldots, b_m\}$. Then for every nonnegative integer n not in B, Alice must have some move on a heap of n stones leading to a position in which the second player wins. That is, every nonnegative integer n (not in B) can be written as $b + p - 1$ for some $b \in B$ and some prime p. In other words, $n - b + 1$ is prime for some b.

Let t be an integer bigger than all of the $b \in B$. Then it is easy to write down t consecutive composite integers: $(t+1)! + 2, \ldots, (t+1)! + t + 1$. Take $n = (t+1)! + t$; then for each $b \in B$, $n - b + 1$ is one of these composite integers, not a prime.

2007 A–3: Assume that we have an ordering of $1, 2, \ldots, 3k+1$ such that no initial subsequence sums to 0 mod 3. If we omit the multiples of 3 from this ordering, then the remaining sequence mod 3 must look like $1, 1, -1, 1, -1, \ldots$ or $-1, -1, 1, -1, 1, \ldots$. Since there is one more integer in the ordering congruent to 1 mod 3 than to -1, the sequence mod 3 must look like $1, 1, -1, 1, -1, \ldots$.

It follows that the ordering satisfies the given condition if and only if the following two conditions hold: the first element in the ordering is not divisible by 3, and the sequence mod 3 (ignoring zeroes) is of the form $1, 1, -1, 1, -1, \ldots$. The two conditions are independent, and the probability of the first is $(2k+1)/(3k+1)$ while the probability of the second is $1/\binom{2k+1}{k}$, since there are $\binom{2k+1}{k}$ ways to order $(k+1)$ 1's and k -1's. Hence the desired probability is the product of these two, or $\frac{k!(k+1)!}{(3k+1)(2k)!}$.

2008 A–2: Barbara wins. The following is a winning strategy.

Pair each entry of the first row with the entry directly below it in the second row. If Alan ever writes a number in one of the first two rows, Barbara writes the same number in the other entry in the pair. If Alan writes a number anywhere other than the first two rows, Barbara does likewise. At the end, the resulting matrix has two identical rows, so its determinant will be zero.

Remark: the archives have multiple solutions for some of these.