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Michelangelo Grigni*
Emory

Leonard Schulman†
Caltech

Umesh Vazirani‡
U. C. Berkeley

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We give a short exposition of new and known results on the "standard method" of identifying a hidden subgroup of a nonabelian group using a quantum computer.

Abbreviated title: Nonabelian Hidden Subgroup

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Contact Address:
Michelangelo Grigni
Emory Univ. Math / CS Dept.
1784 North Decatur Rd.
Atlanta GA 30322
USA

*Contact author. Email: mic@mathcs.emory.edu.
†Email: schulman@cs.caltech.edu.
‡Email: vazirani@cs.berkeley.edu.

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Michelangelo Grigni[*] Emory University
Leonard J. Schulman[†] Caltech
Umesh Vazirani[‡] U. C. Berkeley

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1 Introduction

1.1 Overview

The hidden subgroup problem is the archetypical problem in quantum computation. Although the abelian case is well solved, the nonabelian hidden subgroup problem is a major challenge. The purpose of this paper is to describe the state of knowledge about this problem, focusing on the natural generalization of the technique used successfully in the abelian case, which we will refer to as the "standard method". Indeed, this is arguably the only technique that has been significantly developed to date.

The "hidden subgroup problem" is this. We are given a function $f\colon G \to S$, with the property that f is constant on cosets of an unknown subgroup $H \subseteq G$, and distinct on distinct cosets. Here f is given as an oracle or as an efficient classical program, and S is an arbitrary set. The problem is to determine the hidden subgroup H. (A closely related problem, the "stabilizer problem", was formulated by Kitaev [6].)

The difficulty of the task depends on the type of group G. The abelian case can be effectively computed with a quantum computer by repetition of coset state preparation

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and Fourier sampling — the "standard method" developed by Simon [11] and Shor [10]. In particular this is the heart of Shor's solution of the discrete logarithm and factoring problems.

In the nonabelian case, the standard method is known to be efficient for some "nearly abelian" groups, the dihedral groups [4], although it is only known to be efficient in the information theoretic, rather than computational sense. But more general solutions are unknown; a solution for the symmetric group would yield, for example, the graph automorphism problem (see Section 3.3). In the nonabelian case the transform may again be computed (at least for some groups of interest such as the symmetric group, see for example [1, 3, 8, 2]), but we do not know how to convert similar measurements into a determination of the subgroup. In this paper we give a short summary of these matters. We focus on the statistical information provided by the standard method, rather than group-specific computational issues. We give several "structural" statements, and the following results about the standard method:

1. Normal subgroups: a short derivation of the result of Hallgren, Russell, and Ta-Shma [5] showing that the standard method works efficiently in the case that the hidden subgroup is normal.

2. Involutions: a negative result showing the inefficiency of the weak form of the method in distinguishing between hidden subgroups of size 1 and 2 in arbitrary groups, and particularly in S_n. Part of this result was independently obtained by Hallgren, Russell, and Ta-Shma [5].

3. Random basis: a negative result, showing the inefficiency of the strong form of the method, for determining the hidden subgroup in a general group, when the irreducible representations are computed (in the Fourier transform) in a random basis.

1.2 The Fourier transform and the standard method for hidden subgroup computation

We first recall some basic group representation theory [9]. Given a group G, a matrix representation is a group homomorphism ρ: $G \to GL(d_\rho, \mathbf{C})$, where $GL(d, \mathbf{C})$ is the group of invertible $d \times d$ complex matrices. A finite group G has a finite list of inequivalent irreducible representations $\{\rho\}$, which we henceforth call its *irreps*. Without loss of generality we may assume the irreps are unitary. The sum of the squares of irrep dimensions $\sum_\rho d_\rho^2$ equals $|G|$, the order of the group.

To every group element g we associate a complex vector of dimension $|G|$, indexed by triples ρ, i, j where ρ is an irrep and $1 \leq i, j \leq d_\rho$ indicate an entry of the matrix ρ. The vector associated with g has value $\frac{\sqrt{d_\rho}\rho_{ij}(g)}{\sqrt{|G|}}$ in the ρ, i, j entry.

The Fourier transform over G is the extension of this mapping by linearity to the vector space $\mathbf{C}^G$ of complex linear combinations of group elements. This linear mapping (whose matrix we will denote F) is unitary; this fact is a consequence of the orthogonality relations for group representations.

The trivial representation is the 1-dimensional homomorphism which assigns to every group element the number 1. For a subset S of G, define $|S\rangle = \frac{1}{\sqrt{|S|}} \sum_{g \in S} |g\rangle$ and $\rho(S) = \rho(|S\rangle) = \frac{1}{\sqrt{|S|}} \sum_{g \in S} \rho(g)$. The orthogonality relations imply that $\rho(G)$ is $\sqrt{|G|}$ when ρ is the trivial representation, and a zero matrix otherwise. (As mentioned above, the Fourier transform has a scalar factor $\sqrt{d_\rho/|G|}$, so this corresponds to the fact that the Fourier transform of the unit norm uniform superposition on G, is 1 on the trivial representation and 0 elsewhere.)

$\mathbf{C}^G$ has an additional structure beyond its vector space structure: it is also an algebra over $\mathbf{C}$, using the product which is the extension of the group product by linearity. This structure is preserved by the Fourier transform, simply because each irrep is a group homomorphism. This is what is often known, for abelian groups (where each irrep is 1-dimensional), as the "convolution-multiplication" property of the Fourier transform.

In the "standard method" for the hidden subgroup problem we begin by forming the uniform superposition over a random coset gH of the hidden subgroup H: in other words, we form[1] the uniform distribution over vectors $|gH\rangle$. First suppose that we know g (or at least gH), then we have the pure superposition $|gH\rangle$. We then apply the Fourier transform to this superposition, obtaining the vector

$$\frac{1}{\sqrt{|G||H|}} \sum_{\rho,i,j} \sqrt{d_\rho} \sum_{h \in H} \rho_{ij}(gh) \, |\rho, i, j\rangle \, .$$

[1]To form this mixture of superpositions, we first form the uniform-amplitudes superposition $\frac{1}{\sqrt{|G|}} \sum_{g \in G} |g, 0\rangle$, we then compute f obtaining the superposition $\frac{1}{\sqrt{|G|}} \sum_{g \in G} |g, f(g)\rangle$. We then measure $f(g)$, which determines the coset gH. The result is the superposition $\frac{1}{\sqrt{|H|}} \sum_{h \in H} |gh\rangle$ for a uniformly random g.

By not using $f(g)$ to affect the subsequent computation, we are discarding some potentially useful information. No proposal, however, exists for taking advantage of this information.

This gives rise to the probability distribution

$$P_{gH}(|\rho,i,j\rangle) = \frac{d_\rho}{|G||H|}\left|\sum_{h\in H}\rho_{ij}(gh)\right|^2 = \frac{d_\rho}{|G|}|\rho(gH)_{ij}|^2.$$

Since we actually do not know g, and g is distributed uniformly, we sample ρ, i, j with the probability $P_H(|\rho,i,j\rangle) = \frac{1}{|G|}\sum_{g\in G} P_{gH}(|\rho,i,j\rangle)$.

The success of this method depends on how much statistical information about H is present in this distribution. In particular: do a polynomial number of samples suffice to identify H with high probability? In the following $\chi_\rho(g)$ denotes the *character* of ρ at g, which is simply the trace of $\rho(g)$.

Lemma 1.1 *$\rho(H) = \frac{1}{\sqrt{|H|}}\sum_{h\in H}\rho(h)$ is $\sqrt{|H|}$ times a projection matrix, and* $\mathrm{rank}(\rho(H)) = \frac{1}{|H|}\sum_{h\in H}\chi_\rho(h)$.

Proof: Restricted to H, ρ decomposes into the direct sum of several irreps $\sigma_1, ..., \sigma_k$. $\rho(H)$ is the direct sum of $\sigma_i(H)$; as discussed above $\sigma_i(H)$ is $\sqrt{|H|}$ if σ_i is the trivial representation of H, and zero otherwise. □

A certain amount of information about H is given just by sampling ρ, and ignoring the matrix indices i and j. We refer to this as the "weak form" of the standard method. In the normal case this more limited information is already enough, and in fact no further information is available in the indices. For general subgroups further information is present in the indices, and in the "strong form" of the method, these are sampled as well; we will discuss this issue below. First we show that, when we Fourier sample the unit norm uniform superposition on gH, (i.e. sample from the probability distribution defined by the Fourier transform of this superposition), the probability $P_{gH}(|\rho\rangle)$ of sampling ρ is independent of g.

Lemma 1.2 [2] *The probability of measuring ρ is the same for the uniform superposition on the coset gH (or Hg), as for the superposition on H.*

Proof: $\rho(gH) = \rho(g)\rho(H)$ and $\rho(g)$ is unitary. □

Corollary 1.3 $P_{gH}(|\rho\rangle) = P_H(|\rho\rangle) = \frac{d_\rho}{|G|}\sum_{h\in H}\chi_\rho(h) = \frac{|H|d_\rho}{|G|}\mathrm{rank}(\rho(H))$.

Corollary 1.4 *The probability of sampling ρ is the same for the subgroup H as it is for a conjugate subgroup $g^{-1}Hg$.*

[2] For methods that measure ρ but discard i and j, Lemma 1.2 implies that there is no loss in discarding $f(g)$ as well. In particular we may discard $f(g)$ when G is commutative.

2 Normal H

Recall that in the "standard method" for the hidden subgroup problem we sample from the Fourier transform of the uniform superposition over a random coset gH of the hidden subgroup H. We first show that when we restrict attention to normal subgroups, all the information about H is present in the label of the sampled irrep. By the last lemma, the probability of sampling ρ is independent of the particular coset gH: so we will examine the uniform superposition on H.

Lemma 2.1 *If H is a normal subgroup, $\rho(H)$ is a nonnegative scalar multiple of the identity I, nonzero if and only if $H \subseteq Ker(\rho)$.*

Proof: Let $\sigma_1, ..., \sigma_k$ be the decomposition of ρ for H. We claim that if σ_1 is trivial, so are all the rest.

Let W be the space ρ acts on. Let V be the 1-dimensional subspace of W which σ_1 acts on. Since ρ is irreducible over G, the elements g of G carry V to a set of subspaces spanning W. Since $H = gHg^{-1}$ for every g, each of the images gV is invariant for H. □

By the above lemma, ρ has non-zero probability of being sampled if and only if H is in its kernel. The task of reconstructing H from a sequence of such samples is just that of intersecting the kernels of the sampled irreps. The computational complexity of this task depends upon the underlying group. In this paper we focus on the sample complexity. In order to distinguish H from all other contending subgroups H', it suffices to make the probability of mistaking H' for H, less than inverse in the number of contending subgroups.

How many subgroups can a group G have? No more than $2^{\lg^2 |G|}$: every subgroup has a generating set of size at most $\lfloor \lg |G| \rfloor$, so the number of subgroups is at most $|G|^{\lfloor \lg |G| \rfloor} \leq 2^{\lg^2 |G|}$.

We will show that the variation distance between the distributions on irreps for any two normal subgroups H and H', is bounded from below by a fixed constant. By a large deviation bound, $O(\log^2 |G|)$ repetitions of the sampling process suffice so that the samples uniquely identify the hidden subgroup H.

We begin with the two uniform superpositions $|H\rangle = \frac{1}{\sqrt{|H|}} \sum_{h \in H} |h\rangle$ and $|H'\rangle = \frac{1}{\sqrt{|H'|}} \sum_{h \in H'} |h\rangle$. The L_2 distance between $|H\rangle$ and $|H'\rangle$ is a constant because $H \cap H'$ is at most half the size of one of these groups, wlog H'. So, examining only the elements in $H' - H$, $||H\rangle - |H'\rangle|_2^2 \geq 1/2$. (Exercise: improve the lower bound to $2 - \sqrt{2}$.)

$F|H\rangle$ and $F|H'\rangle$ are the Fourier transforms of $|H\rangle$ and $|H'\rangle$. What we want is a lower bound on the L_1 distance between the distributions arising from measuring these super-

positions (whose probabilities are the norm squares of the amplitudes). We know that $|F|H\rangle - F|H'\rangle|_2 = ||H\rangle - |H'\rangle|_2$ but this alone isn't enough to get an L_1 distance statement, because two superpositions can have nonzero L_2 distance while their distributions have zero L_1 distance. However, this is not a problem for us because of the particular form of $F|H\rangle$ and $F|H'\rangle$: recall from the above lemma that all their amplitudes are nonnegative reals.

We are now in the following situation: we have two nonnegative vectors $F|H\rangle$ and $F|H'\rangle$, which we'll now denote v and w. Each of these is of unit L_2 norm, and $|v-w|_2 \geq \sqrt{2-\sqrt{2}}$. We wish to lower bound $|v^2 - w^2|_1$.

Lemma 2.2 $|v^2 - w^2|_1 \geq |v-w|_2^2$.

Proof: $|v^2 - w^2|_1 = \sum_j |v_j^2 - w_j^2| = \sum_j |v_j - w_j| \cdot |v_j + w_j| \geq \sum_j |v_j - w_j|^2 = |v-w|_2^2$. □

We conclude:

Theorem 2.3 *$O(\log^2 |G|)$ repetitions of Fourier sampling suffice to identify, with high probability, a normal subgroup of a group G.*

3 General H

Up to now we have focussed on the extent to which information about H can be detected from the measuring just the name of the irrep, ρ. Of course we can actually measure more, namely the row i and column j within ρ. It is possible that this contributes substantially to our power. In particular, conjugate subgroups give rise to identical distributions on irreps and so cannot be told apart without measuring the matrix indices within the irreps. In this section, we establish limits on what further information can be obtained from the row and column labels.

3.1 Rows provide no information

In this section we show that there is no point in measuring the row i. (Whether row or column depends on whether the group acts on the left or right. Here we suppose the group acts on the left.) This is because, conditional on measuring ρ and j, the distribution on i is independent of H (actually it's always uniform); we now show how this is due to the fact that in the standard method we average over random cosets gH.

For a particular coset gH, the probability of sampling the entry i,j of ρ is proportional to the norm squared of $\rho(gH)_{ij}$. Thus the probability of sampling entry i,j is

the norm squared of the $|G|$-dimensional vector $(\rho(gH)_{ij})_{g\in G}$ with entries indexed by g. Since $\rho(gH) = \rho(g)\rho(H)$, this vector is a linear combination of the $|G|$-dimensional vectors $(\rho(g)_{ik})_{g\in G}$, with coefficients $\rho(H)_{kj}$. By the orthogonality relations, the d_ρ vectors $(\rho(g)_{ik})_{g\in G}$ are orthonormal, and therefore the norm squared of the $|G|$-dimensional vector $(\rho(gH)_{ij})_{g\in G}$ is equal to the norm squared of the j-th column of $\rho(H)$, and independent of i.

If we keep track of the leading constants, this argument shows:

Theorem 3.1 $P_H(|\rho, i, j\rangle) = \frac{1}{|G|}\,|\rho(H)_j|_2^2$.

3.2 Random basis

The Fourier transform is uniquely defined only up to a change of basis within each irrep; for abelian groups all irreps are one-dimensional so there is no ambiguity in the definition of the transform, but for nonabelian groups there is an arbitrary choice of basis to be made within each irrep. How much statistical information is available by measuring the matrix entries i, j, in addition to the irrep ρ, may in general be basis dependent. In this section, we show that if we choose a random basis for each irrep, then the additional information available is negligible, provided that the subgroup H is sufficiently small and the group G is sufficiently non-abelian. To formalize this we compare the actual distribution $P_H = P_H(|\rho, j\rangle)$ on irreps and columns (we omit the row i thanks to the previous section), with the averaged distribution $\overline{P_H} = \overline{P_H}(|\rho, j\rangle) = \int P_H(|A^{-1}\rho A, j\rangle)dA = \frac{1}{d_\rho}P_H(|\rho\rangle)$, where dA is the Haar measure for unitary matrices. (The last equality follows from the orthogonality relations for representations.) Let $\alpha = \frac{\sqrt{|G|}}{|H|\sqrt{c(G)}}$ where $c(G)$ is the number of conjugacy classes in G.

Theorem 3.2 *Let* $\varepsilon^3 \geq \frac{1}{\alpha}\frac{54}{2-\sqrt{3}}\ln\frac{4|G|}{\delta}$. *Then with probability at least* $1-\delta$ *(over the choice of random basis for the Fourier transform),* $\left|P_H - \overline{P_H}\right|_1 \leq \varepsilon$.

Proof: Given an irrep ρ in a particular basis, the probability of sampling the j-th column of ρ is $\frac{d_\rho}{|G|}\,|\rho(H)_j|_2^2$ (where $\rho(H)_j$ is the j-th column of $\rho(H)$). Suppose we instead choose a random basis for ρ, which we do by replacing ρ by the isomorphic irrep $A^{-1}\rho A$ for A chosen with the Haar distribution in the unitary group. Let A_j be the j-th column of A, which is a vector chosen uniformly from the unit sphere. Then the probability of measuring the j-th column in this modified irrep is $\frac{d_\rho}{|G|}\,|\rho(H)A_j|_2^2$. Our task is to show that for sufficiently large α, this is with high probability close to $\frac{1}{d_\rho}P_H(|\rho\rangle)$.

Let $K = \sum_\rho d_\rho$. We upper bound K by applying the Cauchy-Schwartz inequality to the two vectors $(1)_\rho$ and $(d_\rho)_\rho$. The norm squared of the first vector is simply $c(G)$, the number of irreps of G. Now $K \leq (\sum_\rho 1)^{1/2}(\sum_\rho d_\rho^2)^{1/2} = c(G)^{1/2}|G|^{1/2}$.

To bound the L_1 distance between the vectors P_H and $\overline{P_H}$ we first bound the probability p_T of sampling an irrep ρ with $\mathrm{rank}(\rho(H))$ bounded above by T. After that we will bound the L_1 distance conditional on having sampled an irrep with $\mathrm{rank}(\rho(H)) > T$.

By Corollary 1.3, $P_H(|\rho\rangle) = \frac{|H|d_\rho}{|G|}\mathrm{rank}(\rho(H))$. So $p_T = \sum_{\rho:\ \mathrm{rank}(\rho(H))\leq T} P_H(|\rho\rangle) \leq \frac{|H|T}{|G|}\sum_{\rho:\ \mathrm{rank}(\rho(H))\leq T} d_\rho \leq \frac{|H|T}{|G|}\sum_\rho d_\rho = \frac{|H|TK}{|G|}$ which from the preceding argument is bounded above by $\frac{|H|T\sqrt{c(G)}}{\sqrt{|G|}} = T/\alpha$. We choose $T = \varepsilon\alpha/3$, giving $p_T \leq \varepsilon/3$.

Now for the high rank case, consider the distribution on j conditional on having observed an irrep ρ with $\mathrm{rank}(\rho(H)) > T$. We need to show that with probability at least $1-\delta$ (in terms of the random choice of A), this distribution is within L_1 distance at most $2\varepsilon/3$ of the uniform distribution. We will show slightly more: with probability at least $1-\delta$, for all ρ and all j, $|\rho(H)A_j|_2^2$ deviates from its expectation by at most a $2\varepsilon/3$ fraction. (Since we are concerned here only with fractional error we have suppressed the leading scale factor of the projection $\frac{1}{\sqrt{|H|}}\rho(H)$.)

What we are considering is the following process: a unit vector is chosen uniformly in $\mathbf{C}^{d_\rho}$, then projected onto a fixed subspace of dimension $t > T$; by appropriate change of basis we can without loss of generality suppose that the subspace is spanned by the first t basis vectors of $\mathbf{C}^{d_\rho}$. Let s be the probability that the squared length of the projected vector differs from its expectation t/d_ρ by a fraction greater than than $2\varepsilon/3$. Since we will apply a union bound over all ρ and j, it suffices to show that $s \leq \delta/|G|$. To begin with, note that, due to the isometric correspondence between the unit spheres in $\mathbf{C}^{d_\rho}$ and $\mathbf{R}^{2d_\rho}$, the problem is equivalent to the same problem in real spaces of twice the dimensions, namely projection of the unit sphere in $\mathbf{R}^{2d_\rho}$ onto a $2t$-dimensional subspace. Let M denote the projection matrix; in the appropriate basis it is diagonal, with $2t$ 1's on the diagonal.

We analyze the uniform sampling from the unit sphere indirectly, approximating it by the process of sampling a vector v from the spherically symmetric, $2d_\rho$-dimensional unit variance Gaussian distribution. Let the projection of v be $v' = Mv$. (Note that v' is distributed according to a $2t$-dimensional Gaussian distribution of variance t/d_ρ.) Then $v'/|v|_2$ has the same distribution as $\frac{1}{\sqrt{|H|}}\rho(H)A_j$ (with the understanding that pairs of real coordinates in the first vector form individual complex coordinates in the second). The probability s that $|v'/|v|_2|_2$ deviates from its expectation by fraction $2\varepsilon/3$ is bounded by the sum of the probabilities that $|v|_2$ and $|v'|_2$ deviate from their expectations by fraction $\varepsilon/3$.

We use the following Chernoff bound: if $a_1, ..., a_\tau$ are independent Gaussian random variables each with unit standard deviation, then $P(|\frac{1}{\tau}\sum(a_i^2-1)| > \varepsilon) < 2[(1+\varepsilon)^{1/2}e^{-\varepsilon/2}]^\tau$. For $\varepsilon \leq 2$, $(1+\varepsilon)^{1/2}e^{-\varepsilon/2} \leq \exp(-\varepsilon^2\frac{2-\sqrt{3}}{4})$, and therefore $P(|\frac{1}{\tau}\sum(a_i^2-1)| > \varepsilon) < 2\exp(-\tau\varepsilon^2\frac{2-\sqrt{3}}{4})$. Since this bound is decreasing in τ, and we are applying it with $\tau = 2t > 2T$, we conclude that $s < 4\exp(-2T(\varepsilon/3)^2\frac{2-\sqrt{3}}{4}) = 4\exp(-\alpha\varepsilon^3\frac{2-\sqrt{3}}{54})$.

In order to ensure that $s \leq \delta/|G|$ it suffices therefore that

$$\alpha\varepsilon^3 \geq \frac{54}{2-\sqrt{3}}\ln\frac{4|G|}{\delta}$$

as assumed. □

Corollary 3.3 *With probability at least $1-\delta$ (over the choice of random basis for the Fourier transform), $\Omega((\frac{\alpha}{\log(|G|/\delta)})^{1/3})$ repetitions of Fourier sampling are required in order to achieve constant bias in distinguishing any two (a priori equally probable) conjugate subgroups H and H'.*

Proof: The Hellinger distance $D_H(\vec{p},\vec{q}) = \sum(p^{1/2}-q^{1/2})^2$ between two distributions $\vec{p}$ and $\vec{q}$ is additive across independent samples, and obeys the inequalities $|\vec{p}-\vec{q}|_1^2/4 \leq D_H(\vec{p},\vec{q}) \leq |\vec{p}-\vec{q}|_1$. □

As an example consider the symmetric group $G = S_n$; we know that $c(G) = \exp(\Theta(\sqrt{n}))$. If $|H| \leq |G|^{1/2-\gamma}$ for a fixed $\gamma > 0$ then we need exponentially many samples to gain useful information from j.

3.3 Distinguishing $|H| = 2$ from $|H| = 1$

The graph automorphism problem reduces, via polynomial time reductions [7], to determining the size of the automorphism group in the special case where it is known to be at most 2. So, although sampling irreps cannot distinguish conjugate subgroups, one may still hope that this method is useful for distinguishing subgroups that are not conjugate.

Now consider the special case of distinguishing $H = \{e, s\}$ from $H' = \{e\}$. Let $C(s)$ be the conjugacy class of s.

Theorem 3.4 *The L_1 distance between the distributions on irreps due to Fourier sampling from $|H\rangle$ and $|H'\rangle$, is at most $1/\sqrt{|C(s)|}$.*

Proof: The equation $P_H(|\rho\rangle) = \frac{d_\rho}{|G|}\sum_{h\in H}\chi_\rho(h)$ implies that for H', ρ is sampled with probability $d_\rho^2/|G|$; while for H, ρ is sampled with probability $d_\rho(d_\rho+\chi_\rho(s))/|G|$. So the L_1 distance between the distributions is $\frac{1}{|G|}\sum_\rho d_\rho|\chi_\rho(s)|$.

We upper bound this using the Cauchy-Schwartz inequality and the following equalities:

1. $\sum_\rho d_\rho^2 = |G|$.
2. $|Z(s)| \cdot |C(s)| = |G|$.
3. $\sum_\rho (\chi_\rho(s))^2 = |Z(s)|$.

Here $Z(s)$ is the centralizer of s.

(1) is basic. (2) follows by considering the action of G on itself by conjugation, since under this action $Z(s)$ is the stabilizer of s and $C(s)$ is the orbit of s. (3), which generalizes (1), holds for the following reason. Recall that the unitary character table of G has conjugacy classes labeling columns, irreps labeling rows, and the (ρ, s) entry is $\sqrt{\frac{|C(s)|}{|G|}}\chi_\rho(s)$. Now since each column is unit norm, $1 = \sum_\rho \frac{|C(s)|}{|G|}(\chi_\rho(s))^2$. With (2) this shows (3). Now we can apply Cauchy-Schwartz.

$$\begin{aligned}
\left| |F\,|H\rangle\,|^2 - |F\,|H'\rangle\,|^2 \right|_1 &= \frac{1}{|G|}\sum_\rho d_\rho |\chi_\rho(s)| \\
&\leq \frac{1}{|G|}[\sum_\rho d_\rho^2]^{1/2}[\sum_\rho (\chi_\rho(s))^2]^{1/2} \\
&= (\frac{|Z(s)|}{|G|})^{1/2} = |C(s)|^{-1/2}.
\end{aligned}$$

□

There are examples in which it is challenging to compute s even though the conjugacy class $C(s)$ is known. Observe that in such cases this quantum algorithm has at most a quadratic advantage over the simple probabilistic strategy of checking whether $f(s') = f(e)$ for a random conjugate s'.

We apply Theorem 3.4 in the case that s is an involution in $G = S_n$, i.e. s is a product of some k disjoint transpositions. In this case $|C(s)| = \frac{n!}{2^k k!(n-2k)!}$; as a convenient lower bound on this quantity, count only those conjugates which transpose odd elements with even elements, of which there are $\binom{\lceil n/2 \rceil}{k}\binom{\lfloor n/2 \rfloor}{k}k! \geq k!$. So the L_1 distance between the distributions on irreps is at most $(k!)^{-1/2}$. In the graph automorphism application k can be proportional to n, in which case this is exponentially small.

Finally we combine this bound with Theorem 3.2. Note that $\alpha = \frac{\sqrt{n!}}{2 \cdot 2^{\Theta(\sqrt{n})}} \in \exp(\frac{1}{2}n \ln n - O(n))$, so:

Corollary 3.5 *If we apply the standard method, using a random basis, to the graph automorphism problem, then with probability at least $1 - \delta$ the L_1 distance between the Fourier sampling distribution given that the automorphism group is trivial, and the Fourier sampling distribution given that the automorphism group is of size 2 and contains an involution with k transpositions, is at most $\exp(-\frac{1}{6}n \ln n + O(n)) \log^{\frac{1}{3}} \frac{1}{\delta} + (k!)^{-1/2}$.*

Just as in Corollary 3.3, the upper bound on L_1 distance implies a lower bound on the number of samples which must be collected in order to distinguish the hypotheses reliably.

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