1 Basic definitions and lemmas

We will be dealing only with biconnected graphs. Let G be a graph. A *correct embedding* of G is a planar map embedding of G in which no point is met by more than four countries. Our goal is to design an algorithm which given G, constructs a correct embedding of G if one exists, and reports "failure" otherwise. Since checking the correctness of an embedding of G can be done in polynomial time, we can *assume* that G has a correct embedding and only need to show how to find such an embedding. So, in the rest discussion of this paper, *we assume that G has a correct embedding.*

Clearly, every correct embedding of G can be transformed into another correct embedding $\mathcal{E}$ satisfying the following condition $(*)$:

$(*)$ No two distinct points p_1 and p_2 satisfy that p_1 is met by at least three countries and every country meeting p_1 also meets p_2.

Thus, in the sequel, a correct embedding always means one that satisfies $(*)$ above.

We call vertices in G *countries*. We use $V(G)$ (or $E(G)$, respectively) to denote the set of countries (respectively, edges) in G. For a country $c \in V(G)$, $N_G(c)$ denotes the set of countries adjacent to c in G, and $deg_G(c) = |N_G(c)|$. Let $U \subseteq V(G)$ and $F \subseteq E(G)$.

- For two distinct countries c and d in G, $dist_G(c,d)$ denotes the distance between c and d in G.
- $N_G(U) = \cup_{c \in U} N_G(c)$, and $G[U]$ denotes the graph (U, E_U), where $E_U = \{\{c,d\} \in E(G) \mid c \text{ and } d \text{ are in } U\}$.
- $G - U - F$ denotes the graph obtained from G by deleting the edges in F and the countries (together with the edges incident to them) in U.
- For a subset W of U, $\mathcal{C}^G_{U,F}(W) = \{c \in V(G) - U \mid W = N_G(K) \cap U$, where K is the connected component of $G - U - F$ containing $c\}$.
- A *correct layout* of $G[U]$ is a planar map embedding of $G[U]$ that can be extended to a correct embedding of G. Let $\mathcal{L}$ be a correct layout of $G[U]$. A country d is a *neighbor* of another country c in $\mathcal{L}$ if they touch each other at two or more points in $\mathcal{L}$. For a country c and a neighbor d of c in $\mathcal{L}$, $H_{\mathcal{L}}[c,d]$ denotes the union of all the holes enclosed by c and d in $\mathcal{L}$. That is, c and d may meet each other inside $H_{\mathcal{L}}[c,d]$ but may never meet each other outside $H_{\mathcal{L}}[c,d]$. Possibly, $H_{\mathcal{L}}[c,d]$ is empty. For simplicity, we often call $H_{\mathcal{L}}[c,d]$ a hole in $\mathcal{L}$. When $\mathcal{L}$ is clear from the context, we will write $H[c,d]$ for $H_{\mathcal{L}}[c,d]$. We will frequently use figures to demonstrate how $\mathcal{L}$ looks like. In these figures, some important points will be given names. In particular, for each country c and each neighbor d of c in $\mathcal{L}$, we will represent $H_{\mathcal{L}}[c,d]$ by a single hole and give names to the two points (on the boundary of $H_{\mathcal{L}}[c,d]$) where c and d touch each other. A figure demonstrating $\mathcal{L}$ is said to be *explicit* if all the points with names in it are distinct except that for one or more holes $H_{\mathcal{L}}[c,d]$, the two points with names on the boundary of $H_{\mathcal{L}}[c,d]$ may be equal. When a figure demonstrating $\mathcal{L}$ is explicit, we call a correct embedding $\mathcal{E}$ of G *a maximum extension* of $\mathcal{L}$ if (1) $\mathcal{E}$ is an extension of $\mathcal{L}$ and (2) among all extensions of $\mathcal{L}$, the number of countries of

$G - U$ embedded in $H[c, d]$ is maximum for *every* country $c \in U$ and *every* neighbor d of c in $\mathcal{L}$. Note that there may exist two or more maximum extensions of $\mathcal{L}$.

- We say that a layout $\mathcal{L}$ of $G[U]$ can be *transformed* to another layout $\mathcal{L}'$ of $G[U]$ if whenever $\mathcal{L}$ is correct, so is $\mathcal{L}'$.

When U or F is empty, we drop it from the notations $G - U - F$ and $\mathcal{C}^G_{U,F}(W)$.

A maximal clique of size k is denoted by MC_k. It is easy to see that G has no MC_k with $k \geq 7$. Let l be a positive integer. We say that two maximal cliques C and C' are *l-sharing* if $|C \cap C'| = l$.

Definition 1.1 A *correct 4-point* is a cyclicly ordered list $\langle c_0, \ldots, c_3, c_0 \rangle$ of four countries in G such that G has a correct embedding in which (1) the four countries c_0 through c_3 meet at the same point (say, p) in this order and (2) whenever c_0 and c_2 (or c_1 and c_3, respectively) together with two other countries d' and d'' meet at a point $q \neq p$, the cyclic order of the four countries around q is c_0, d', c_2, d'', c_0 (respectively, c_1, d', c_3, d'', c_1).

Definition 1.2 *Removing a correct 4-point* $P = \langle c_0, \ldots, c_3, c_0 \rangle$ from G is the operation of modifying G as follows: Delete the two edges $\{c_0, c_2\}$ and $\{c_1, c_3\}$ from G and then adding to G one new country c and the four edges in $\{\{c, c_i\} \mid 0 \leq i \leq 3\}$.

Lemma 1.3 Let G' be the graph obtained from G by removing a correct 4-point $P = \langle c_0, \cdots, c_3, c_0 \rangle$. Then, the following hold:

(1) G' has a correct embedding.

(2) G' has fewer MC_4's if G' has neither MC_5 nor MC_6.

(3) Given an arbitrary correct embedding of G', we can construct a correct embedding of G in linear time.

Proof. We only prove (1); (2) and (3) are obvious. Let $\mathcal{E}$ be a correct embedding of G satisfying the two conditions (1) and (2) in Definition 1.1. Without loss of generality, we can assume that no point in $\mathcal{E}$ is met by exactly three points. We modify $\mathcal{E}$ as follows. For the point p where the four countries c_0 through c_3 meet, we amplify p to a disk and embed the new country c in this disk (see Figure 1.1). For each point $q \neq p$ where c_0 and c_2 together with other two countries meet, we amplify q to a disk while keeping both c_0 and c_2 meet the other two countries (see Figure 1.2). Note that if c_0 and c_2 meet at a point other than p, then c_1 and c_3 cannot meet at a point other than p. Clearly, the modified $\mathcal{E}$ is a correct embedding of G'. ∎

It is not difficult to see that every MC_5 must be layouted as a "pizza with crust". Thus, in every correct layout of an MC_5 C, there is a point met by exactly four countries in C. This motivated the following definition:

Definition 1.4 A *correct center* of an MC_5 C in G is a cyclicly ordered list $\langle c_0, \ldots, c_3, c_0 \rangle$ of four countries in C such that C has a correct layout in which the four countries c_0 through c_3 meet at the same point in this order.

Fact 1 Let C be an MC_5 in G. Then, every correct center of C is a correct 4-point in G. Moreover, after removing it from G, G has fewer MC_5's if G has no MC_6.

Proof. Let $C = \{c_0, \ldots, c_4\}$. Let $\mathcal{L}$ be a correct layout of C witnessing that $\langle c_0, \ldots, c_3, c_0\rangle$ is a correct center of C. Since we assume $(*)$ above, the four countries c_0 through c_3 meet at exactly one point p in $\mathcal{L}$ and their order around p is c_0, …, c_3, c_0. It is easy to see that if c_0 and c_2 (or c_1 and c_3, respectively) meet at a point $q \neq p$ in $\mathcal{L}$, then c_4 and exactly one $c_i \in \{c_1, c_3\}$ (respectively, $c_i \in \{c_0, c_2\}$) must also meet q and the cyclic order around q is c_0, c_4, c_2, c_i, c_0 (respectively, c_1, c_4, c_3, c_i, c_1). The second assertion in the fact is obvious. ∎

Lemma 1.5 Let $k \geq 2$ be an integer. Let $\langle c_0, \ldots, c_{k-1}, c_0\rangle$ be a cyclicly ordered list of k countries such that every two consecutive countries are adjacent in G. Let G' be the graph obtained from G by adding $k+1$ new countries $d_0, \ldots, d_{k-1}, e$ and the $3k$ edges in the set $\{\{e, d_i\} \mid 0 \leq i \leq k-1\} \cup \{\{d_i, c_j\} \mid 0 \leq i \leq k-1 \text{ and } j = i \text{ or } i+1 \bmod k\}$. Then, G' has a correct embedding iff G has a correct embedding in which only the k countries $c_0, \ldots, c_{k-1}$ are on the boundary of the infinite face. Moreover, given a correct embedding of G', we can construct in polynomial time a correct embedding of G in which only the k countries $c_0, \ldots, c_{k-1}$ are on the boundary of the infinite face.
Proof. Since the sufficient condition is obvious, we only prove the necessary condition. Let $\mathcal{E}'$ be a correct embedding of G'. Since $\{d_i, c_i, c_{i+1}\}$ is an MC_3 in G' for every $0 \leq i \leq k-1$, we can assume that there are k distinct points $p_0, \ldots, p_{k-1}$ such that each p_i is met by exactly d_i, c_i and c_{i+1}. Now, by planarity, the k countries d_0 through d_{k-1} touching e in $\mathcal{E}'$ must appear around e in the cyclic order d_0, …, d_{k-1}, d_0 (see Figure 1.3). Recall that G is biconnected. Thus, each area A_i in Figure 1.3 is a hole. Now, it is easy to modify $\mathcal{E}'$ to obtain a correct embedding of G in which only the k countries $c_0, \ldots, c_{k-1}$ are on the boundary of the infinite face. ∎

2 Outline of the algorithm

The basic idea is to remove correct 4-points from a given graph G whenever possible. If this is impossible, we seek to divide G into smaller ones such that G has a correct embedding iff all the smaller graphs have a correct embedding. The bird-view of the algorithm is as follows.

Input: A graph $G = (V, E)$.

Output: An embedding of G that is correct whenever G has a correct embedding.

1. If some maximal clique in G has size larger than 6, then report "failure" and halt.

2. While G has an MC_6 C, perform the following:

- **2.1.** Find a layout of C that is correct whenever G has a correct embedding.
- **2.2.** Use the layout found in step 2.1 to decompose the graph into smaller ones such that if G has a correct embedding, then so do all the smaller graphs and a correct embedding of G can be constructed in polynomial time when given a correct embedding for each of the smaller graphs. [*Comment:* The decomposition will be done in such a way that each of the smaller graph has fewer MC_6's than G and the total number of countries in the smaller graphs is greater than the number of countries in G by a constant.]

2.3. Recursively construct an embedding of each of the smaller graphs, and then combine the returned embeddings to obtain an embedding of G that is correct whenever G has a correct embedding.

3. While G has an MC_5 C, perform the following:

3.1. *Either* find a correct center of C *or* decompose the graph into smaller ones such that if G has a correct embedding, then so do all the smaller graphs and a correct embedding of G can be constructed in polynomial time when given a correct embedding for each of the smaller graphs. [*Comment:* The decomposition will be done in such a way that each of the smaller graph has fewer MC_5's than G and the total number of countries in the smaller graphs is greater than the number of countries in G by a constant.]

3.2. If a correct center P was found in step 3.1, then recursively construct an embedding of the graph obtained from G by removing P, and then use the returned embedding to construct an embedding of G that is correct whenever G has a correct embedding.

3.3. If G was decomposed into smaller graphs in step 3.1, recursively construct an embedding of each of the smaller graphs, and then combine the returned embeddings to obtain an embedding of G that is correct whenever G has a correct embedding.

4. Label every MC_4 of G "non-pizza".

5. While G has an MC_4 with label "non-pizza", perform the following:

5.1. Label C "pizza" iff C has no correct non-pizza layout.

5.2. If the label of C is still "non-pizza", then *either* find one or more correct 4-points *or* decompose the graph into smaller ones such that if G has a correct embedding, then so do all the smaller graphs and a correct embedding of G can be constructed in polynomial time when given a correct embedding for each of the smaller graphs. [*Comment:* The decomposition will be done in such a way that each of the smaller graph has fewer MC_4's than G and the total number of countries in the smaller graphs is greater than the number of countries in G by a constant.]

5.3. If one or more correct 4-points were found in step 5.2, then recursively construct an embedding of the graph obtained from G by removing the 4-points, and then use the returned embedding to construct an embedding of G that is correct whenever G has a correct embedding.

5.3. If G was decomposed into smaller graphs in step 5.1, then recursively construct an embedding of each of the smaller graphs, and then combine the returned embeddings to obtain an embedding of G that is correct whenever G has a correct embedding.

6. Construct a bipartite graph $H_1 = (X, Y, E_1)$ as follows. X is the set of all countries still remaining in G and Y is the set of all MC_4's still remaining in G. For each MC_4 $y \in Y$ and each country $x \in y$, there is an edge connecting x and y in H_1.

7. Construct a bipartite graph H_2 from H_1 as follows. For each pair of adjacent countries x_1 and x_2 still remaining in G, if there is no MC_4 $y \in Y$ such that $\{x_1, x_2\} \subset y$, then add a new node $z_{1,2}$ and two new edges $\{x_1, z_{1,2}\}$ and $\{x_2, z_{1,2}\}$ to H_1. [*Comment:* We can show that every MC_3 in G can be layouted as a hamantaschen.]

8. If H_2 is not a planar graph, then report "failure". Otherwise, compute a planar embedding of H_2, and use it to construct a correct embedding of G.

3 Removing maximal cliques of size 6

Recall that G is assumed to have a correct embedding. Let $C = \{c_1, \cdots, c_6\}$ be an MC_6 in G. It is not difficult to see that every correct layout of C can be transformed into another correct layout of the form shown in Figure 3.1. In Figure 3.1, $\pi = (c'_1, \ldots, c'_6)$ is a permutation of $(c_1, \ldots, c_6)$. Moreover, Figure 3.1 is explicit.

Theorem 3.1 Let $S_1 = \{(1,2), (3,4), (5,6)\}$, $S_2 = \{(2,3), (2,5), (3,5), (1,4), (1,6), (4,6)\}$, and $T = \{(2,3,5), (1,4,6)\}$. Then, for every permutation $\pi = (c'_1, \ldots, c'_6)$ of $(c_1, \ldots, c_6)$, the layout shown in Figure 3.1 is correct iff the family $\mathcal{F} = \{\mathcal{C}^G_C(\{c'_i, c'_j\}) \mid (i,j) \in S_1 \cup S_2\} \cup \{\mathcal{C}^G_C(\{c'_i, c'_j, c'_k\}) \mid (i,j,k) \in T\}$ is a partition of $V(G) - C$.
Proof. Let $\mathcal{E}_{\pi,\max}$ be a maximum extension of the layout of C shown in Figure 3.1 such that for every pair $(i,j) \in S_2$, whenever two countries d_1 and d_2 in $G - C$ meet the point $p_{i,j}$ in $\mathcal{E}_{\pi,\max}$, the cyclic order of the four countries around $p_{i,j}$ is c'_i, c'_j, d_1, d_2, c'_i.

Lemma 3.2 For every pair $(i,j) \in S_1 \cup S_2$, $\mathcal{C}^G_C(\{c'_i, c'_j\})$ is exactly the set of countries of $G - C$ embedded in $H[c'_i, c'_j]$ by $\mathcal{E}_{\pi,\max}$. Consequently, for each triple $(i,j,k) \in T$, $\mathcal{C}^G_C(\{c'_i, c'_j, c'_k\})$ is exactly the set of countries of $G - C$ embedded in the hole $A_{i,j,k}$ by $\mathcal{E}_{\pi,\max}$.
Proof. Consider the hole $H[c'_2, c'_3]$. Clearly, only those countries in $\mathcal{C}^G_C(\{c'_2, c'_3\})$ can be embedded in $H[c'_2, c'_3]$ by $\mathcal{E}_{\pi,\max}$. Towards a contradiction, assume that some country d in $G - C$ is not embedded in $H[c'_2, c'_3]$ by $\mathcal{E}_{\pi,\max}$. Then, the whole connected component of $G - C$ containing d must be embedded in the hole $A_{2,3,5}$. However, we can move this component from $A_{2,3,5}$ to $H[c'_2, c'_3]$. This contradicts the choice of $\mathcal{E}_{\pi,\max}$. Therefore, $\mathcal{C}^G_C(\{c'_2, c'_3\})$ is exactly the set of countries of $G - C$ embedded in $H[c'_2, c'_3]$ by $\mathcal{E}_{\pi,\max}$. ∎

By Lemma 3.2 and Figure 3.1, the family $\mathcal{F}$ is a partition of $V(G) - C$. To show Theorem 3.1, it suffices to show that whenever C has a correct layout $\mathcal{L}$ of the form shown in Figure 3.1, $\mathcal{E}_{\pi,\max}$ must be a correct layout of C, too. We may assume that $\mathcal{L} = \mathcal{E}_{\pi',\max}$ for some permutation π' of $(c_1, \ldots, c_6)$. It remains to show that no matter what π' is, $\mathcal{E}_{\pi',\max}$ can be transformed to $\mathcal{E}_{\pi,\max}$ without altering the correctness. For example, suppose that $\mathcal{L}$ corresponds to the permutation $\pi' = (c'_1, c'_5, c'_3, c'_4, c'_2, c'_6)$. By the assumption in the theorem, both $\mathcal{C}^G_C(\{c'_1, c'_5\})$ and $\mathcal{C}^G_C(\{c'_2, c'_6\})$ must be empty. Thus, by Lemma 3.2, we can assume that $\mathcal{L}$ is of the form shown in Figure 3.2. Figure 3.2 is explicit. By this figure, it is clear that $\mathcal{E}_{\pi',\max}$ can be transformed to a correct embedding of G which is an extension of the layout shown in Figure 3.1. ∎

By Theorem 3.1, we can compute a correct layout of C in linear time. Suppose that we have found that the layout shown in Figure 3.1 is correct. Then, the following (1) and (2) both hold:

(1) For every pair $(i,j) \in S_1 \cup S_2$, the subgraph of G induced by $\{c'_i, c'_j\} \cup \mathcal{C}^G_C(\{c'_i, c'_j\})$ has a correct embedding in which only the two countries c'_i and c'_j are on the boundary of the infinite face.

(2) For every triple $(i,j,k) \in T$, the subgraph of G induced by $\{c'_i, c'_j, c'_k\} \cup \mathcal{C}^G_C(\{c'_i, c'_j, c'_k\})$ has a correct embedding in which only the three countries c'_i, c'_j and c'_k are on the boundary of the infinite face.

By Lemma 1.5 and its proof, constructing the correct embeddings stated in (1) and (2) above can be reduced in polynomial time to constructing correct embeddings of certain graphs such that each of the graphs has fewer MC_6's than G and the total number of countries in the graphs is larger than that in G by a constant.

4 Removing maximal cliques of size 5

Recall that G is assumed to have a correct embedding. By the arguments in § 3, we may further assume that G has no MC_6. Let $C = \{c_1, \ldots, c_5\}$ be an MC_5 in G. We try to find a correct center of C. If we fail eventually, we will be able to decompose G into smaller graphs. It is easy to see that every correct layout of C can be transformed to another correct layout of one of the five forms shown in Figure 4.1. In each form, $(c'_1, \ldots, c'_5)$ is a permutation of $(c_1, \ldots, c_5)$. All the five figures are explicit. Hereafter, a correct layout of C always means one that is of one of the forms in Figure 4.1. For each layout in Figure 4.1, we call the cyclicly ordered list $(c'_1, \ldots, c'_4, c'_1)$ the *center* of the layout. We call a country $c \in C$ a *correct crust* of C if there is a correct layout of C whose center does not include c. ¿From Figure 4.1, it is not difficult to see the following facts:

Fact 2 There is at most two other distinct MC_5's 4-sharing with C.

Fact 3 If W is a subset of C such that $|W| \geq 3$ and $\mathcal{C}^G_C(W) \neq \emptyset$, then no country in $C - W$ is a correct crust of C.

Fact 4 Let C' be a maximal clique in G. Then, if $|C' \cap C| \geq 3$, no country in $C - C'$ is a correct crust of C. Moreover, if $|C' \cap C| = 2$, then in every correct layout of C whose center includes both the two countries in $C' \cap C$, the two countries must appear around the center consecutively.

Theorem 4.1 Assume that C has a correct layout of the form shown in Figure 4.1(a). Let F be the set of the edges $\{d_1, d_2\}$ in $G - C$ such that $\{d_1, d_2, c_i, c_j\}$ is an MC_4 in G for some $\{c_i, c_j\} \subset C$. Let $S = \{(1,2),(1,4),(2,3),(3,4),(1,5),...,(4,5)\}$ and $T = \{(1,2,5),(2,3,5),(3,4,5),(1,4,5)\}$. Then, for every permutation $(c'_1, \ldots, c'_5)$ of $(c_1, \ldots, c_5)$, the layout shown in Figure 4.1(a) is a correct layout of C iff the family $\mathcal{F} = \{\mathcal{C}^G_{C,F}(\{c'_i, c'_j\}) \mid (i,j) \in S\} \cup \{\mathcal{C}^G_{C,F}(\{c'_i, c'_j, c'_k\}) \mid (i,j,k) \in T\}$ is a partition of $V(G) - C$.

Proof. Let $\pi = (c'_1, \ldots, c'_5)$ be a permutation of $(c_1, \ldots, c_5)$. Let $\mathcal{E}_{\pi,\max}$ be the maximum extension of the layout in Figure 4.1(a).

Lemma 4.2 For every pair $(i,j) \in S$, $\mathcal{C}^G_{C,F}(\{c'_i, c'_j\})$ is exactly the set of countries of $G - C$ embedded in $H[c'_i, c'_j]$ by $\mathcal{E}_{\pi,\max}$. Consequently, for each triple $(i,j,k) \in T$, $\mathcal{C}^G_{C,F}(\{c'_i, c'_j, c'_k\})$

is exactly the set of countries of $G - C$ embedded in $A_{i,j,k}$ by $\mathcal{E}_{\pi,\max}$, where $A_{i,j,k}$ is the hole enclosed by the countries c'_i, c'_j, and c'_k.

Proof. For example, consider the hole $H[c'_1, c'_5]$. Clearly, only those countries in $\mathcal{C}^G_{C,F}(\{c'_1, c'_5\})$ can be embedded in $H[c'_1, c'_5]$. Towards a contradiction, assume that some country in $\mathcal{C}^G_{C,F}(\{c'_1, c'_5\})$ is not embedded in $H[c'_1, c'_5]$ by $\mathcal{E}_{\pi,\max}$. Let K be the connected component of $G - C - F$ containing this country. Then, by Figure 4.1(a), all the countries in K must be embedded either in the hole $A_{1,2,5}$ or in the hole $A_{1,4,5}$. We assume that they are embedded in $A_{1,2,5}$ by $\mathcal{E}_{\pi,\max}$; the case where they are embedded in $A_{1,4,5}$ by $\mathcal{E}_{\pi,\max}$ is similar. Figure 4.2(a) shows the layout of $G[K \cup \{c'_1, c'_2, c'_5\}]$ obtained by deleting from $\mathcal{E}_{\pi,\max}$ all the countries not in $K \cup \{c'_1, c'_2, c'_5\}$. Figure 4.2(a) is explicit except that possibly $r_1 = r_2$, $r_3 = r_4$, $r_1 = p_{1,5}$, $r_4 = p_{1,5}$, $r_2 = p_{1,5}$, or $r_3 = p_{1,5}$.

Case 1: Neither $r_1 = p_{1,5}$ nor $r_4 = p_{1,5}$. Imagine that we start at r_1 and traverse clockwise along the boundary of K until reaching the point r_4. If we meet no country other than c'_1 and c'_5 during the traversing, then clearly we can modify $\mathcal{E}_{\pi,\max}$ as shown in Figure 4.2(b), contradicting the choice of $\mathcal{E}_{\pi,\max}$. Otherwise, let d be an arbitrary country met by us during the traversing. Let e be an arbitrary country in K touched by d in $\mathcal{E}_{\pi,\max}$. Since d touches e but they belong to different connected components of $G - C - F$, $\{c'_1, c'_5, d, e\}$ must be an MC_4 in G and there is no country $f \notin \{c'_1, c'_5\}$ such that f, e, and d meet each other at the same point in $\mathcal{E}_{\pi,\max}$. So, by Figure 4.2(a), d must be unique and we may assume that there is exactly one point t where d and e meet each other (see Figure 4.2(c)). Obviously, we can modify $\mathcal{E}_{\pi,\max}$ so that c'_1 and c'_5 also meet at t (see Figure 4.2(d)) without altering its correctness, contradicting the choice of $\mathcal{E}_{\pi,\max}$.

Case 2: Some $r_i \in \{r_1, r_4\}$ equals $p_{1,5}$. If either no country of $G - (K \cup C)$ embedded in the hole $A_{1,2,5}$ by $\mathcal{E}_{\pi,\max}$ touches K or some country of $G - (K \cup C)$ embedded in the hole $A_{1,2,5}$ by $\mathcal{E}_{\pi,\max}$ touches K but does not meet $p_{1,5}$, then similarly to case 1, we can modify $\mathcal{E}_{\pi,\max}$ so that K is embedded in the hole $A_{1,5}$, contradicting the choice of $\mathcal{E}_{\pi,\max}$. Otherwise, exactly one country d embedded in the hole $A_{1,2,5}$ by $\mathcal{E}_{\pi,\max}$ touches K, d meets the point $p_{1,5}$, and we can assume that $p_{1,5} \neq q_{1,5}$. Since d, K, c'_1, and c'_5 meet each other at $p_{1,5}$, no country embedded in the hole $A_{1,5}$ by $\mathcal{E}_{\pi,\max}$ can meet $p_{1,5}$ and hence we can move the whole K into $A_{1,5}$, a contradiction. ∎

By Lemma 4.2, it is easy to see that the family $\mathcal{F}$ is a partition of $V(G) - C$. To show Theorem 4.1, it suffices to show that whenever C has a correct layout $\mathcal{L}$ of the form shown in Figure 4.1(a), $\mathcal{E}_{\pi,\max}$ must be a correct layout of C, too. We may assume that $\mathcal{L} = \mathcal{E}_{\pi',\max}$ for some permutation π' of $(c_1, \ldots, c_5)$. It remains to show that no matter what π' is, $\mathcal{E}_{\pi',\max}$ can be transformed to $\mathcal{E}_{\pi,\max}$ without altering the correctness. For example, suppose that $\pi' = (c'_1, c'_3, c'_2, c'_5, c'_4)$. Since $\mathcal{F}$ is a partition of $V(G) - C$, $\mathcal{C}^G_{C,F}(\{c'_1, c'_3\}) = \emptyset$, $\mathcal{C}^G_{C,F}(\{c'_1, c'_3, c'_4\}) = \emptyset$, $\mathcal{C}^G_{C,F}(\{c'_2, c'_3, c'_4\}) = \emptyset$, $\mathcal{C}^G_{C,F}(\{c'_2, c'_4\}) = \emptyset$, and $\mathcal{C}^G_{C,F}(\{c'_2, c'_4, c'_5\}) = \emptyset$. Hence, we can assume that $\mathcal{L}$ is as shown in Figure 4.2(e) which explicit. By Figure 4.2(e), it is clear that $\mathcal{E}_{\pi',\max}$ can be transformed to a correct embedding of G which is an extension of the layout shown in Figure 4.1(a). ∎

By Theorem 4.1, if we know that C has a correct layout of the form shown in Figure 4.1(a), then a correct layout of C can be computed in polynomial time. Our strategy is trying to figure out the cases where C has a correct layout of the form shown in Figure 4.1(a). For example, when G has neither MC_5 4-sharing with C nor MC_4 3-sharing with C, it is not difficult to see that C must have a correct layout of the form shown in Figure 4.1(a). So, in the sequel, we assume that G has an MC$_5$ 4-sharing with C or an MC$_4$ 3-sharing with C.

4.1 The case having two 4-sharing MC$_5$'s

Throughout this subsection, we assume that G has two distinct MC$_5$'s that are 4-sharing with C. Let them be $C_1 = \{c_2, c_3, c_4, c_5, c_6\}$ and $C_2 = \{c_1, c_2, c_3, c_5, c_7\}$. Let $U = C \cup \{c_6, c_7\}$.

Let us first suppose that c_6 and c_7 are adjacent in G. Then by Figure 4.1, every correct layout of $C \cup \{c_6, c_7\}$ can be transformed to another of one of the two forms shown in Figure 4.1.1. In both forms, (c'_2, c'_3, c'_5) is a permutation of (c_2, c_3, c_5). Both Figure 4.1.1(a) and Figure 4.1.1(b) are explicit. For convenience, let $c'_1 = c_1$, $c'_4 = c_4$, $c'_6 = c_6$, and $c'_7 = c_7$. Similarly to Theorem 3.1, we can prove the following two theorems:

Theorem 4.3
Let $S = \{(1,2),(1,4),(1,5),(2,3),(2,7),(3,4),(3,6),(4,5),(5,6),(5,7),(6,7)\}$ and $T = \{(1,4,5),(5,6,7)\}$. Then, for every permutation (c'_2, c'_3, c'_5) of (c_2, c_3, c_5), the layout in Figure 4.1.1(a) is correct iff the family $\mathcal{F} = \{\mathcal{C}^G_C(\{c'_i, c'_j\}) \mid (i,j) \in S\} \cup \{\mathcal{C}^G_C(\{c'_i, c'_j, c'_k\}) \mid (i,j,k) \in T\}$ is a partition of $V(G) - C$.

Theorem 4.4
Let $S = \{(1,2),(1,3),(1,7),(2,4),(2,6),(2,7),(3,4),(3,5),(4,6),(5,6),(5,7)\}$ and $T = \{(1,2,7),(2,4,6)\}$. Then, for every permutation (c'_2, c'_3, c'_5) of (c_2, c_3, c_5), the layout in Figure 4.1.1(b) is correct iff the family $\mathcal{F} = \{\mathcal{C}^G_C(\{c'_i, c'_j\}) \mid (i,j) \in S\} \cup \{\mathcal{C}^G_C(\{c'_i, c'_j, c'_k\}) \mid (i,j,k) \in T\}$ is a partition of $V(G) - C$.

By Theorem 4.3 and Theorem 4.4, we can compute a correct layout of C in linear time. Suppose that we have found that the layout in Figure 4.1.1(a) is correct. Then, the following (1) and (2) both hold:

(1) For every pair $(i,j) \in S$, the subgraph of G induced by $\{c'_i, c'_j\} \cup \mathcal{C}^G_U(\{c'_i, c'_j\})$ has a correct embedding in which only the two countries c'_i and c'_j are on the boundary of the infinite face.

(2) For every triple $(i,j,k) \in T$, the subgraph of G induced by $\{c'_i, c'_j, c'_k\} \cup \mathcal{C}^G_U(\{c'_i, c'_j, c'_k\})$ has a correct embedding in which only the three countries c'_i, c'_j and c'_k are on the boundary of the infinite face.

By Lemma 1.5 and its proof, constructing the correct embeddings stated in (1) and (2) above can be reduced in polynomial time to constructing correct embeddings of certain graphs such that each of the graphs has fewer MC$_5$'s than G and the total number of countries in the graphs is larger than that in G by a constant.

If we have found that the layout in Figure 4.1.1(b) is correct, we can decompose G into smaller graphs in a similar way.

Now, suppose that c_6 and c_7 are not adjacent in G. Then by Figure 4.1, every correct layout of $C \cup \{c_6, c_7\}$ can be transformed to another of one of the two forms shown in

Figure 4.1.2. In both forms, (c'_2, c'_3, c'_5) is a permutation of (c_2, c_3, c_5). Figure 4.1.2(a) is explicit except that possibly $p_{2,6} = p_{2,4}$, $p_{5,6} = q_{2,6}$, $p_{3,7} = p_{1,3}$, or $p_{5,7} = q_{3,7}$. Figure 4.1.2(b) is explicit except that possibly $p_{2,6} = p_{2,4}$, $p_{5,6} = q_{2,6}$, $p_{2,6} = p_{2,5}$, $p_{2,7} = p_{1,2}$, $p_{5,7} = q_{2,7}$, or $p_{2,7} = q_{2,5}$. Actually, we only need to know which one of c_2, c_3, and c_5 is a correct crust of C, because this information always gives us a correct center of C. Once we know a correct center of C, then we can remove it from G by Fact 1 and Lemma 1.3.

By Figure 4.1.2, there is no connected component K in $G - U$ such that $\{c_2, c_3, c_5\} \subseteq N_G(K)$. Thus, for every connected component K in $G - U$, $|N_G(K) \cap \{c_2, c_3, c_5\}| \leq 2$. Let us first suppose that there is no connected component K in $G - U$ such that $|N_G(K) \cap \{c_2, c_3, c_5\}| = 2$. Then, by Figure 4.1.2, it is easy to see that every correct layout of $G[U]$ can be transformed to another of one of the two forms in Figure 4.1.3. In both forms, (c'_2, c'_3, c'_5) is a permutation of (c_2, c_3, c_5). Both Figure 4.1.3(a) and Figure 4.1.3(b) are explicit. Similarly to Theorem 3.1, we can prove the following two theorems:

Theorem 4.5 Let $S = \{(1,2), (1,3), (1,7), (2,4), (2,6), (3,4), (3,7), (4,6), (5,6), (5,7)\}$ and $T = \{(1,3,7), (2,4,6)\}$. Then, for every permutation (c'_2, c'_3, c'_5) of (c_2, c_3, c_5), the layout in Figure 4.1.3(a) is correct iff the family $\{\mathcal{C}^G_C(\{c'_i, c'_j\}) \mid (i,j) \in S\} \cup \{\mathcal{C}^G_C(\{c'_i, c'_j, c'_k\}) \mid (i,j,k) \in T\}$ is a partition of $V(G) - C$.

Theorem 4.6 Let $S = \{(1,2), (1,3), (1,7), (2,4), (2,6), (2,7), (3,4), (4,6), (5,6), (5,7)\}$ and $T = \{(1,2,7), (2,4,6)\}$. Then, for every permutation (c'_2, c'_3, c'_5) of (c_2, c_3, c_5), the layout in Figure 4.1.3(b) is correct iff the family $\{\mathcal{C}^G_C(\{c'_i, c'_j\}) \mid (i,j) \in S\} \cup \{\mathcal{C}^G_C(\{c'_i, c'_j, c'_k\}) \mid (i,j,k) \in T\}$ is a partition of $V(G) - C$.

Now, suppose that there is a connected component K in $G - U$ such that $|N_G(K) \cap \{c_2, c_3, c_5\}| = 2$. We assume that $N_G(K) \cap \{c_2, c_3, c_5\} = \{c_2, c_5\}$; the other cases are similar. Then, c_3 is not a correct crust of C and we only need to distinguish c_2 and c_5. By Figure 4.1.2, if there is another connected component K' in $G - U$ such that (1) $\{c_3, c_i\} \subseteq N_G(K')$ for some $c_i \in \{c_2, c_5\}$ or (2) $N_G(K') \cap \{c_1, c_4\} \neq \emptyset$ and $|N_G(K') \cap \{c_2, c_5\}| = 1$, then we know which one of c_2 and c_5 must be a correct crust of C and we are done. So, we may assume that the following (1) and (2) hold:

(1) There is no connected component K in $G - U$ such that $\{c_3, c_2\} \subseteq N_G(K')$ or $\{c_3, c_5\} \subseteq N_G(K')$.

(2) There is no connected component K in $G - U$ such that $N_G(K') \cap \{c_1, c_4\} \neq \emptyset$ and $|N_G(K') \cap \{c_2, c_5\}| = 1$.

Then, every correct layout of $G[U]$ can be transformed to another of one of the four forms in Figure 4.1.4. In all the forms, (c'_2, c'_5) is a permutation of (c_2, c_5). Figure 4.1.4(a) is explicit. Figure 4.1.4(b) is explicit except that possibly $p_{1,2} = p_{2,7}$ or $p_{5,7} = q_{2,7}$. Figure 4.1.4(c) is explicit except that possibly $p_{2,4} = p_{2,6}$ or $p_{5,6} = q_{2,6}$. Figure 4.1.4(d) is explicit except that possibly $p_{2,6} = p_{2,4}$, $p_{5,6} = q_{2,6}$, $p_{2,6} = p_{2,5}$, $p_{2,7} = p_{1,2}$, $p_{5,7} = q_{2,7}$, or $p_{2,7} = q_{2,5}$. For $1 \leq i \leq 7$, let $\mathcal{K}_i$ be the class of the connected components K in $G - U$ with $c_i \in N_G(K)$. Possibly, $\mathcal{K}_1$, ..., $\mathcal{K}_7$ are not disjoint.

Case 1: $\mathcal{K}_3 \cap \mathcal{K}_6 \neq \emptyset$ and $\mathcal{K}_3 \cap \mathcal{K}_7 \neq \emptyset$. Then, $G[U]$ has a correct layout of the form in Figure 4.1.4(a). It is easy to prove that for every permutation (c'_2, c'_5) of (c_2, c_5), the layout in Figure 4.1.4(a) is correct iff $\mathcal{C}^G_U(\{c'_2, c_6\}) = \mathcal{C}^G_U(\{c'_2, c_7\}) = \emptyset$.

Case 2: $\mathcal{K}_3 \cap \mathcal{K}_6 = \emptyset$ and $\mathcal{K}_3 \cap \mathcal{K}_7 \neq \emptyset$. Then, every correct layout of $G[U]$ can be transformed to another of the form in Figure 4.1.4(c). By Figure 4.1.4(c), if there is exactly one $c_i \in \{c_2, c_5\}$ such that $\mathcal{K}_7 \cap \mathcal{K}_i \neq \emptyset$ or $\mathcal{K}_4 \cap \mathcal{K}_i \neq \emptyset$, then we know which of c_2 and c_5 must be a correct layout of C. Moreover, if $\mathcal{K}_4 \cap \mathcal{K}_2 = \mathcal{K}_4 \cap \mathcal{K}_5 = \emptyset$, then every correct layout of $G[U]$ can be transformed to another of the form in Figure 4.1.5. In Figure 4.1.5, (c'_2, c'_5) is a permutation of (c_2, c_5) and Figure 4.1.5 is explicit. We can prove that for every permutation (c'_2, c'_5) of (c_2, c_5), the layout in Figure 4.1.5 is a correct layout of $G[U]$ iff $\mathcal{C}_U^G(\{c'_2, c_7\}) = \emptyset$. Hence, we may assume that $\mathcal{K}_7 \cap \mathcal{K}_2 = \mathcal{K}_7 \cap \mathcal{K}_5 = \emptyset$, $\mathcal{K}_4 \cap \mathcal{K}_2 \neq \emptyset$, and $\mathcal{K}_4 \cap \mathcal{K}_5 \neq \emptyset$. Let $G' = G - \{c_1, c_3, c_6\} - \{\{c_4, c_2\}, \{c_4, c_5\}\}$. By Figure 4.1.4(c), the distance from c_4 to c'_2 in G' does not exceed that from c_4 to c'_5 in G'. Thus, if the distances from c_4 to c_2 and c_5 in G' are not equal, then we know which one of c_2 and c_5 must be a correct crust of C. So, we further assume that the distances from c_4 to c_2 and c_5 in G' are equal. Let the distance be l.

Case 2.1: $l \geq 3$. Then by Figure 4.1.4(c), there is a country $c_8 \notin U$ such that $\{c_2, c_5, c_6, c_8\}$ is an MC_4 in G and every correct layout of $G[U \cup \{c_8\}]$ can be transformed to another of the form in Figure 4.1.6. Figure 4.1.6 is explicit. Let $U_8 = U \cup \{c_8\}$. Then, there is no connected component K in $G - U_8$ such that $\{c_4, c'_5\} \subseteq N_G(K)$ or $\{c_8, c'_5\} \subseteq N_G(K)$. Thus, we may assume that there is no connected component K in $G - U_8$ such that $N_G(K) \cap \{c_2, c_5\} \neq \emptyset$ and $N_G(K) \cap \{c_4, c_8\} \neq \emptyset$; otherwise, we know which one of c_2 and c_5 must be a correct crust of C. Then, Figure 4.1.6 can be assumed to be explicit. Moreover, by considering a maximum extension of the layout of $G[U_8]$ in Figure 4.1.6, we can prove that for every permutation (c'_2, c'_5) of (c_2, c_5), the layout in Figure 4.1.6 is correct iff $\mathcal{C}_{U_8}^G(\{c'_2, c_6\}) = \emptyset$.

Case 2.2: $l = 2$. Then by Figure 4.1.4(c), there is a country $c_8 \notin \{c_1, \ldots, c_7\}$ such that $C_3 = \{c_2, c_4, c_5, c_6, c_8\}$ is an MC_5 in G. Moreover, every correct layout of $G[U \cup \{c_8\}]$ can be transformed to another of one of the two forms in Figure 4.1.7. Figure 4.1.7(a) is explicit except that possibly $p_{5,8} = p_{6,8}$ or $q_{5,8} = p_{2,8}$. Figure 4.1.7(b) is explicit except that possibly $p_{4,8} = p_{2,4}$ or $q_{4,8} = p_{6,8}$. Let $U_8 = U \cup \{c_8\}$.

Case 2.2.1: There is some connected component K in $G - U_8$ such that $\{c_4, c_8\} \subseteq N_G(K)$. Then, $G[U_8]$ has a correct layout of the form in Figure 4.1.7(b). By Figure 4.1.7(b), there is no connected component K in $G - U_8$ such that $\{c_4, c'_5\} \subseteq N_G(K)$ or $\{c_8, c'_5\} \subseteq N_G(K)$. So, we may assume that there is no connected component K in $G - U_8$ such that $N_G(K) \cap \{c_2, c_5\} \neq \emptyset$ and $N_G(K) \cap \{c_4, c_8\} \neq \emptyset$; otherwise, we know which one of c_2 and c_5 must be used a correct crust of C. Then, Figure 4.1.7(b) can be assumed to be explicit. Moreover, by considering a maximum extension of the layout of $G[U_8]$ in Figure 4.1.7(b), we can prove that for every permutation (c'_2, c'_5) of (c_2, c_5), the layout in Figure 4.1.7(b) is correct iff $\mathcal{C}_{U_8}^G(\{c'_2, c_6\}) = \emptyset$.

Case 2.2.2: There is no connected component K in $G - U_8$ such that $\{c_4, c_8\} \subseteq N_G(K)$. Then, every correct layout of $G[U_8]$ of the form in Figure 4.1.7(b) can be transformed to another of the form in Figure 4.1.7(a), and hence $G[U_8]$ always has a correct layout of the latter form. Let G' be the graph obtained from G by removing c_1, c_3, c_7, and all the countries that can reach c_1, c_3, or c_7 in the graph $G - U_8$. Note that C_3 is an MC_5 in G'. Figure 4.1.8(a) shows a layout of C_3 that can be extended to a correct layout of G'. In Figure 4.1.8(a), (c'_2, c'_5) is a permutation of (c_2, c_5). Figure 4.1.8(a) is explicit except that possibly $p_{5,8} = p_{6,8}$ or $q_{5,8} = p_{2,8}$. By Figure 4.1.7(a), in order to compute a correct embedding of G, it suffices to perform the following two steps:

(1) For each subset W among $\{c_1, c_3\}$, $\{c_1, c_7\}$, $\{c_3, c_7\}$, $\{c_1, c_3, c_7\}$, and $\{c_3, c_4\}$, compute a correct embedding of $G[W \cup \mathcal{C}_{U_8}^G(W)]$ in which only the countries in W are on the

boundary of the infinite face.

(2) Compute a correct embedding of G' in which either c_2 or c_5 is used as the crust of the MC_5 C_3 and the center of the MC_5 C_3 is as shown in Figure 4.1.8(a).

Step (1) can be done recursively according to Lemma 1.5. Next, consider step (2). If $|N_{G'}(K) \cap C_3| \leq 2$ for every connected component K in $G' - C_3$, then Figure 4.1.7(a) can be assumed to be explicit and we can prove that for every permutation (c'_2, c'_5) of (c_2, c_5), the layout in Figure 4.1.7(a) is correct iff $\mathcal{C}^G_{U_8}(\{c'_2, c_6\}) = \emptyset$. So, we may assume that there is a connected component K in $G' - C_3$ such that $|N_{G'}(K) \cap C_3| \geq 3$. Then, c_4 is not a correct crust of the MC_5 C_3 in G' by Fact 3. Now, we add to G' three new countries d_1, d_2, d_3 and seven new edges $\{c_8, d_1\}$, $\{d_1, c_6\}$, $\{c_6, d_2\}$, $\{d_2, c_4\}$, $\{d_3, c_2\}$, $\{d_3, c_5\}$, and $\{d_3, c_4\}$. Obviously, G' still has a correct embedding (see Figure 4.1.8(b)) and according to Fact 4, every correct layout of the MC_5 C_3 in G' must satisfy that either c_2 or c_5 is used as the crust of the MC_5 C_3 and the center of the MC_5 C_3 is as shown in Figure 4.1.8(a). Note that G' has fewer MC_5's than G. So, to perform step (2), it suffices to compute a correct embedding of G' recursively.

Case 3: $\mathcal{K}_3 \cap \mathcal{K}_6 \neq \emptyset$ and $\mathcal{K}_3 \cap \mathcal{K}_7 = \emptyset$. This case is similar to Case 2.

Case 4: $\mathcal{K}_3 \cap \mathcal{K}_6 = \emptyset$ and $\mathcal{K}_3 \cap \mathcal{K}_7 = \emptyset$. Then, every correct layout of $G[U]$ can be transformed to another of the form in Figure 4.1.4(d). Recall that there is no connected component K in $G - U$ such that $N_G(K) \cap \{c_1, c_4\} \neq \emptyset$ and $|N_G(K) \cap \{c_2, c_5\}| = 1$. Thus, one of the following three subcases must occur:

Case 4.1: There is no connected component K in $G - U$ such that $N_G(K) \cap \{c_1, c_4\} \neq \emptyset$ and $N_G(K) \cap \{c_2, c_5\} \neq \emptyset$. Then, every correct layout of $G[U]$ can be transformed to another of one of the three forms in Figure 4.1.9. In each form, (c'_2, c'_5) is a permutation of (c_2, c_5). All the three figures in Figure 4.1.9 are explicit. For each of the three layouts, by considering its maximum extension, we can prove that exchanging the positions of c'_2 and c'_5 in it does not alter its correctness. Thus, both c_2 and c_5 are a correct crust of C.

Case 4.2: For exactly one $c_i \in \{c_1, c_4\}$, there is no connected component K in $G - U$ such that $c_i \in N_G(K)$ and $N_G(K) \cap \{c_2, c_5\} \neq \emptyset$. We assume that $c_i = c_1$; the case where $c_i = c_4$ is similar. Then by Figure 4.1.4(b), every correct layout of $G[U]$ can be transformed to another of the form in Figure 4.1.10. Figure 4.1.10 is explicit except that possibly $p_{2,6} = p_{2,4}$, $p_{5,6} = q_{2,6}$, $p_{2,6} = p_{2,5}$, or $p_{2,7} = q_{2,5}$. By this figure, it is not hard to see that in the graph $G' = G - \{c_1, c_3, c_6\} - \{\{c_2, c_4\}, \{c_4, c_5\}\}$, the distance from c_4 to c'_2 does not exceed that from c_4 to c'_5. Thus, if the distances from c_4 to c_2 and c_5 in G' are not equal, then we know which of c_2 and c_5 must be a correct crust of C. So, we assume that the distances from c_4 to c_2 and c_5 in G' are equal. Let l be the distance.

Case 4.2.1: $l \geq 3$. Then by Figure 4.1.10, there is a country $c_8 \notin \{c_1, \ldots, c_7\}$ such that $\{c_2, c_5, c_6, c_8\}$ is an MC_4 in G and every correct layout of $G[U \cup \{c_8\}]$ can be transformed to another of the form in Figure 4.1.11. Figure 4.1.11 is explicit. Let $U_8 = U \cup \{c_8\}$. By Figure 4.1.11, there is no connected component K in $G - U_8$ such that $\{c_4, c'_5\} \subseteq N_G(K)$ or $\{c_8, c'_5\} \subseteq N_G(K)$. Thus, we may assume that there is no connected component K in $G - U_8$ such that $N_G(K) \cap \{c_2, c_5\} \neq \emptyset$ and $N_G(K) \cap \{c_4, c_8\} \neq \emptyset$; otherwise, we know which one of c_2 and c_5 must be a correct crust of C. Then, Figure 4.1.11(a) can be assumed to be explicit. Moreover, by considering a maximum extension of the layout of $G[U_8]$ in Figure 4.1.11, we can prove that for every permutation (c'_2, c'_5) of (c_2, c_5), the layout in Figure 4.1.11 is correct iff $\mathcal{C}^G_{U_8}(\{c'_2, c_6\}) = \emptyset$.

Case 4.2.2: $l = 2$. Then by Figure 4.1.10, there is a country $c_8 \notin U_8$ such that

$C_3 = \{c_2, c_4, c_5, c_6, c_8\}$ is an MC_5 in G. Let $U_8 = U \cup \{c_8\}$. If $\{c_7, c_8\} \in E(G)$, then $G[U_8]$ has a correct layout of the form in Figure 4.1.12. Figure 4.1.12 is explicit and (c'_2, c'_5) is a permutation of (c_2, c_5) in it. It is easy to prove that the layout of $G[U_8]$ in Figure 4.1.12 is correct iff $\mathcal{C}^G_{U_8}(\{c'_2, c_6\}) = \mathcal{C}^G_{U_8}(\{c'_2, c_6, c_8\}) = \mathcal{C}^G_{U_8}(\{c'_2, c_7\}) = \emptyset$. So, we assume that $\{c_7, c_8\} \notin E(G)$. Then, every correct layout of $G[U \cup \{c_8\}]$ can be transformed to another of one of the two forms in Figure 4.1.13. Figure 4.1.13(a) is explicit except that possibly $p_{2,4} = p_{4,8}$ or $p_{6,8} = q_{4,8}$. Figure 4.1.13(b) is explicit except that possibly $p_{6,8} = p_{5,8}$, $p_{2,8} = q_{5,8}$, $p_{2,8} = p_{2,5}$, or $p_{2,7} = q_{2,5}$. Let $U_8 = U \cup \{c_8\}$.

Case 4.2.2.1: There is some connected component K in $G - U_8$ such that $\{c_4, c_8\} \subseteq N_G(K)$. Then, $G[U_8]$ has a correct layout of the form in Figure 4.1.13(a). By this figure, there is no connected component K in $G - U_8$ such that $\{c_4, c'_5\} \subseteq N_G(K)$ or $\{c_8, c'_5\} \subseteq N_G(K)$. Thus, we may assume that there is no connected component K in $G - U_8$ such that $N_G(K) \cap \{c_2, c_5\} \neq \emptyset$ and $N_G(K) \cap \{c_4, c_8\} \neq \emptyset$; otherwise, we know which one of c_2 and c_5 must be a correct crust of C. Then, Figure 4.1.13(a) can be assumed to be explicit. Moreover, by considering a maximum extension of the layout of $G[U_8]$ in Figure 4.1.13(a), we can prove that for every permutation (c'_2, c'_5) of (c_2, c_5), the layout in Figure 4.1.13(a) is correct iff $\mathcal{C}^G_{U_8}(\{c'_2, c_6\}) = \emptyset$.

Case 4.2.2.2: There is no connected component K in $G - U_8$ such that $\{c_4, c_8\} \subseteq N_G(K)$. Then, $G[U_8]$ has a correct layout of the form in Figure 4.1.13(b). If there is no $d \notin U_8$ such that $|N_G(d) \cap \{c_2, c_5, c_6, c_8\}| \geq 3$, then either Figure 4.1.13(b) can be assumed to be explicit or $G[U_8]$ has a correct layout of the form in Figure 4.1.14 which is explicit. In both cases, by considering the maximum extensions of the two layouts of $G[U_8]$ in Figure 4.1.13(b) and Figure 4.1.14, we can prove that c_5 is a correct crust of the MC_5 C iff $\mathcal{C}^G_{U_8,F_8}(\{c_2, c_6\}) = \mathcal{C}^G_{U_8,F_8}(\{c_2, c_6, c_8\}) = \emptyset$, where F_8 is the set of the edges $\{d_1, d_2\}$ in $G - U_8$ such that $\{d_1, d_2, c_i, c_j\}$ is an MC_4 in G for some $\{c_i, c_j\} \subset U_8$. Thus, we may assume that there is some $d \notin U_8$ such that $|N_G(d) \cap \{c_2, c_5, c_6, c_8\}| \geq 3$.

Let G' be the graph obtained from G by (i) removing c_1, c_3, and all the countries that can reach c_1 or c_3 in the graph $G - U_8$ and by (ii) adding five new countries $d_1, \ldots, d_5$ and appropriate new edges so that $\{c_8, d_1, c_6\}$, $\{c_6, d_2, c_4\}$, $\{c_2, c_5, d_3, c_7\}$, $\{c_2, c_5, c_4, d_3\}$, $\{c_2, d_4, c_7\}$, and $\{c_5, d_5, c_7\}$ are maximal cliques in G'. Figure 4.1.15 shows a layout of $G'[\{c_2, c_4, \ldots, c_8, d_1, \ldots, d_5\}]$ that can be extended to a correct embedding of G'. In Figure 4.1.15, (c'_2, c'_5) is a permutation of (c_2, c_5) and (d'_4, d'_5) is a permutation of (d_4, d_5). Figure 4.1.15 is explicit except that possibly $p_{6,8} = p_{5,8}$, $p_{2,8} = q_{5,8}$, $p_{2,8} = p_{2,5}$, or $p_{2,7} = q_{2,5}$. Note that $C_3 = \{c_2, c_4, c_5, c_6, c_8\}$ is an MC_5 in G'. By Figure 4.1.13(b), in order to compute a correct embedding of G, it suffices to perform the following two steps:

(1) For each subset W among $\{c_1, c_3\}$, $\{c_1, c_7\}$, and $\{c_3, c_4\}$, compute a correct embedding of $G[W \cup \mathcal{C}^G_{U_8}(W)]$ in which only the countries in W are on the boundary of the infinite face.

(2) Compute a correct embedding of G' in which

(2.1) either c_2 or c_5 is used as the crust of the MC_5 C_3,

(2.2) the center of the MC_5 C_3 is as shown in Figure 4.1.15, and

(2.3) the four countries c_2, c_4, c_5, c_7 appear around d_3 in the cyclic order c_2, c_4, c_5, c_7, c_2.

Step (1) can be done recursively according to Lemma 1.5. Next, consider step (2). Recall that there is some $d \notin U_8$ such that $|N_G(d) \cap \{c_2, c_5, c_6, c_8\}| \geq 3$. This d must be contained in G'. Thus, by Fact 3 and Fact 4, every correct embedding of G' must satisfy the conditions (2.1) and (2.2). It is also obvious that every correct embedding of G' satisfying (2.1) and (2.2) must satisfy the condition (2.3), too. So, to perform step (2), it suffices to compute an arbitrary correct embedding of G' recursively.

Case 4.3: For each $c_i \in \{c_1, c_4\}$, there is a connected component K in $G - U$ such that $\{c_i, c_2, c_5\} \subseteq N_G(K)$. Then, every correct layout of $G[U]$ can be transformed to another of the form in Figure 4.1.4(d). Then, in the graph $G' = G - \{c_3, c_6, c_7\} - \{\{c_1, c_2\}, \{c_1, c_5\}, \{c_4, c_2\}, \{c_4, c_5\}, \{c_4, c_1\}\}$, the distance from each $c_i \in \{c_1, c_4\}$ to c'_2 does not exceed the distance from c_i to c'_5. Thus, we may assume that for each $c_i \in \{c_1, c_4\}$, the distances from $c_i \in \{c_1, c_4\}$ to c_2 and c_5 in G' are equal; otherwise, we know which one of c_2 and c_5 must be a correct crust. Let l_1 be the distance from c_1 to c_2 (and hence to c_5) in G'. Let l_4 be the distance from c_4 to c_2 (and hence to c_5) in G'.

Case 4.3.1: $l_1 = l_4 = 2$. Then, there must exist two distinct countries c_8 and c_9 not in U such that $\{c_2, c_5, c_1, c_7, c_8\}$ and $\{c_2, c_5, c_4, c_6, c_9\}$ are MC_5's in G. Let $U_9 = \{c_1, \ldots, c_9\}$. If $\{c_8, c_9\} \in E(G)$, then $G[U_9]$ has a correct layout of the form in Figure 4.1.16. Figure 4.1.16 is explicit and (c'_2, c'_5) is a permutation of (c_2, c_5) in it. It is easy to prove that the layout of $G[U_9]$ in Figure 4.1.16 is correct iff $\mathcal{C}^G_{U_9}(W) = \emptyset$ for each W among the sets $\{c'_2, c_6\})$, $\{c'_2, c_6, c_9\}$, $\{c'_2, c_7\})$, and $\{c'_2, c_7, c_8\}$. So, we assume that $\{c_8, c_9\} \notin E(G)$. Then, by Figure 4.1.4(d), every correct layout of $G[U_9]$ can be transformed to another of one of the four forms in Figure 4.1.17. In each of the forms, (c'_2, c'_5) is a permutation of (c_2, c_5). Figure 4.1.17(a) is explicit except that possibly $p_{1,8} = p_{1,2}$, $p_{7,8} = q_{1,8}$, $p_{4,9} = p_{2,4}$, or $q_{4,9} = p_{6,9}$. Figure 4.1.17(b) is explicit except that possibly $p_{2,8} = q_{5,8}$, $p_{7,8} = p_{5,8}$, $p_{4,9} = p_{2,4}$, or $q_{4,9} = p_{6,9}$. Figure 4.1.17(c) is explicit except that possibly $p_{1,8} = p_{1,2}$, $p_{7,8} = q_{1,8}$, $p_{2,9} = q_{5,9}$, or $p_{5,9} = p_{6,9}$. Figure 4.1.17(d) is explicit except that possibly $p_{2,8} = q_{5,8}$, $p_{7,8} = p_{5,8}$, $p_{2,9} = q_{5,9}$, $p_{5,9} = p_{6,9}$, $q_{2,5} = p_{5,9}$, or $p_{2,5} = p_{5,8}$.

Case 4.3.1.1: There are connected components K_1 and K_2 in $G - U_9$ such that $\{c_1, c_8\} \subseteq N_G(K_1)$ and $\{c_4, c_9\} \subseteq N_G(K_2)$. Then, $G[U_9]$ has a correct layout of the form in Figure 4.1.17(a). By Figure 4.1.17(a), there is no connected component K in $G - U_9$ such that either $c'_5 \in N_G(K)$ and $N_G(K) \cap \{c_1, c_4, c_8, c_9\} \neq \emptyset$ or $c'_2 \in N_G(K)$ and $N_G(K) \cap \{c_6, c_7\} \neq \emptyset$. Thus, we may assume that there is no connected component K in $G - U_9$ such that $N_G(K) \cap \{c_2, c_5\} \neq \emptyset$ and $N_G(K) \cap \{c_1, c_4, c_6, \ldots, c_9\} \neq \emptyset$; otherwise, we know which one of c_2 and c_5 must be a correct crust of C. But then, exchanging the positions of c'_2 and c'_5 in Figure 4.1.17(a) does not alter the correctness of the layout, and both c_2 and c_5 are a correct crust of C.

Case 4.3.1.2: There is no connected component K_1 in $G - U_9$ with $\{c_1, c_8\} \subseteq N_G(K_1)$ but there is a connected component K_2 in $G - U_9$ with $\{c_4, c_9\} \subseteq N_G(K_2)$. Then, $G[U_9]$ has a correct layout of the form in Figure 4.1.17(b). By Figure 4.1.17(b), there is no connected component K in $G - U_9$ such that either $c'_5 \in N_G(K)$ and $N_G(K) \cap \{c_4, c_9\} \neq \emptyset$ or $\{c'_2, c_6\} \subseteq N_G(K)$. Thus, we may assume that there is no connected component K in $G - U_9$ such that $N_G(K) \cap \{c_2, c_5\} \neq \emptyset$ and $N_G(K) \cap \{c_4, c_9, c_6\} \neq \emptyset$; otherwise, we know which one of c_2 and c_5 must be a correct crust of C. Then, by Figure 4.1.17(b), every correct layout of $G[U_9]$ can be transfromed to another of the form in Figure 4.1.18(a). In Figure 4.1.18(a), (c'_2, c'_5) is a permutation of (c_2, c_5). Figure 4.1.18(a) is explicit except that possibly $p_{2,8} = q_{5,8}$ or $p_{7,8} = p_{5,8}$. If there is no $d \notin U_9$ such that $|N_G(d) \cap \{c_2, c_5, c_7, c_8\}| \geq 3$, then Figure 4.1.18(a) can be assumed to be explicit and by considering the maximum extension of the layout of $G[U_9]$ in Figure 4.1.18(a), we can prove that c_5 is a correct crust of the MC_5 C iff $\mathcal{C}^G_{U_9,F_9}(\{c_2, c_7\}) = \mathcal{C}^G_{U_9,F_9}(\{c_2, c_7, c_8\}) = \emptyset$, where F_9 is the set of the edges

$\{d_1, d_2\}$ in $G - U_9$ such that $\{d_1, d_2, c_i, c_j\}$ is an MC_4 in G for some $\{c_i, c_j\} \subset U_9$. Thus, we may assume that there is some $d \notin U_9$ such that $|N_G(d) \cap \{c_2, c_5, c_7, c_8\}| \geq 3$.

Let G'' be the graph obtained from G by (i) removing c_3, c_4, c_6, c_9, and all the countries that can reach c_3, c_4, c_6, or c_9 in the graph $G - U_9$ and by (ii) adding three new countries d_1, d_2, d_3 and appropriate new edges so that $\{c_8, d_1, c_7\}$, $\{c_7, d_2, c_1\}$, and $\{c_2, c_5, d_3, c_1\}$ are maximal cliques in G''. Figure 4.1.18(b) shows a layout of $G''[\{c_1, c_2, c_5, c_7, c_8, d_1, d_2, d_3\}]$ that can be extended to a correct layout of G''. Figure 4.1.18(b) is explicit except that possibly $p_{2,8} = q_{5,8}$ or $p_{7,8} = p_{5,8}$. Note that $C_3 = \{c_1, c_2, c_5, c_7, c_8\}$ is an MC_5 in G''. By Figure 4.1.18(a), in order to compute a correct embedding of G, it suffices to perform the following two steps:

(1) For each subset W among $\{c_1, c_3\}$, $\{c_3, c_4\}$, $\{c_4, c_6\}$, $\{c_4, c_9\}$, $\{c_6, c_9\}$, and $\{c_4, c_6, c_9\}$, compute a correct embedding of $G[W \cup \mathcal{C}^G_{U_9}(W)]$ in which only the countries in W are on the boundary of the infinite face.

(2) Compute a correct embedding of G'' in which

(2.1) either c_2 or c_5 is used as the crust of the MC_5 C_3 and

(2.2) the center of the MC_5 C_3 is as shown in Figure 4.1.18(b).

Step (1) can be done recursively according to Lemma 1.5. Next, consider step (2). Recall that there is some $d \notin U_9$ such that $|N_G(d) \cap \{c_2, c_5, c_7, c_8\}| \geq 3$. This d must be contained in G''. Thus, by Fact 3 and Fact 4, every correct embedding of G'' must satisfy the conditions (2.1) and (2.2). So, to perform step (2), it suffices to compute an arbitrary correct embedding of G'' recursively.

Case 4.3.1.3: There is a connected component K_1 in $G - U_9$ with $\{c_1, c_8\} \subseteq N_G(K_1)$ but there is no connected component K_2 in $G - U_9$ with $\{c_4, c_9\} \subseteq N_G(K_2)$. This case is similar to Case 4.3.1.2.

Case 4.3.1.4: There are no connected components K_1 and K_2 in $G - U_9$ such that $\{c_1, c_8\} \subseteq N_G(K_1)$ and $\{c_4, c_9\} \subseteq N_G(K_2)$. Then, $G[U_9]$ has a correct layout of the form in Figure 4.1.17(d). Let G'' be the graph obtained from G by (i) removing c_1, c_4, and all the countries that can reach c_1 or c_4 in the graph $G - U_9$ and by (ii) adding two new countries d_1 and d_2 and appropriate new edges so that $\{c_6, d_1, c_3\}$, $\{c_3, d_2, c_7\}$, and $\{c_2, c_5, c_6, c_7, c_3\}$ are maximal cliques in G''. Figure 4.1.19 shows a layout of $G''[\{c_2, c_3, c_5, \ldots, c_9, d_1, d_2\}]$ that can be extended to a correct layout of G''. Figure 4.1.19 is explicit except that possibly $p_{2,8} = q_{5,8}$, $p_{7,8} = p_{5,8}$, $p_{2,9} = q_{5,9}$, $p_{5,9} = p_{6,9}$, $q_{2,5} = p_{5,9}$, or $p_{2,5} = p_{5,8}$. Note that $C_3 = \{c_2, c_5, c_6, c_7, c_3\}$ is an MC_5 in G''. By Figure 4.1.17(d), in order to compute a correct embedding of G, it suffices to perform the following two steps:

(1) For each subset W among $\{c_1, c_3\}$, $\{c_1, c_7\}$, $\{c_3, c_4\}$, and $\{c_4, c_6\}$, compute a correct embedding of $G[W \cup \mathcal{C}^G_{U_9}(W)]$ in which only the countries in W are on the boundary of the infinite face.

(2) Compute a correct embedding of G'' in which

(2.1) either c_2 or c_5 is used as the crust of the MC_5 C_3 and

(2.2) the center of the MC_5 C_3 is as shown in Figure 4.1.19.

Step (1) can be done recursively according to Lemma 1.5. By Fact 3 and Fact 4, every correct embedding of G'' must satisfy the conditions (2.1) and (2.2). So, to perform step (2), it suffices to compute an arbitrary correct embedding of G'' recursively.

Case 4.3.2: $l_1 = 2$ and $l_4 \geq 3$. Then, there must exist two distinct countries c_8 and c_9 not in U such that $\{c_2, c_5, c_1, c_7, c_8\}$ and $\{c_2, c_5, c_6, c_9\}$ are an MC_5 and an MC_4 in G, respectively. Let $U_9 = \{c_1, \ldots, c_9\}$. By Figure 4.1.4(d), every correct layout of $G[U_9]$

can be transformed to another of one of the two forms in Figure 4.1.20. In both of the forms, (c'_2, c'_5) is a permutation of (c_2, c_5). Figure 4.1.20(a) is explicit except that possibly $p_{1,2} = p_{1,8}$ or $p_{7,8} = q_{1,8}$. Figure 4.1.20(b) is explicit except that possibly $p_{2,8} = q_{5,8}$ or $p_{7,8} = p_{5,8}$.

Case 4.3.2.1: There is a connected component in $G - U_9$ such that $\{c_1, c_8\} \subseteq N_G(K)$. Then, $G[U_9]$ has a correct layout of the form in Figure 4.1.20(a). Similarly to Case 4.3.1.1, we can decide which of c_2 and c_5 must be a correct crust of C.

Case 4.3.2.2: There is no connected component in $G - U_9$ such that $\{c_1, c_8\} \subseteq N_G(K)$. Then, $G[U_9]$ has a correct layout of the form in Figure 4.1.20(b). This case can be dealt with similarly to Case 4.3.1.2.

Case 4.3.3: $l_1 \geq 3$ and $l_4 = 2$. This case is similar to Case 4.3.2.

Case 4.3.4: $l_1 \geq 3$ and $l_4 \geq 3$. Then, there must exist two distinct countries c_8 and c_9 not in U such that $\{c_2, c_5, c_7, c_8\}$ and $\{c_2, c_5, c_6, c_9\}$ are MC_4's in G, respectively. Let $U_9 = \{c_1, \ldots, c_9\}$. By Figure 4.1.4(d), every correct layout of $G[U_9]$ can be transformed to another of the form in Figure 4.1.21. Figure 4.1.21 is explicit. Similarly to case 4.3.1.1, we can figure out which one of c_2 and c_5 is a correct crust of C.

4.2 The case having one 4-sharing MC_5

By the discussions in § 4.1, we can assume that no MC_5 in G is 4-sharing with two other MC_5's. Throughout this subsection, we assume that G has an MC_5 that is 4-sharing with C. Let it be $C_1 = \{c_1, c_2, c_3, c_5, c_6\}$. Let $U = \{c_1, \ldots, c_6\}$. Let F be the set of the edges $\{d_1, d_2\}$ in $G - U$ such that $\{d_1, d_2, c_i, c_j\}$ is an MC_4 in G for some $\{c_i, c_j\} \subset U$. By Figure 4.1, every correct layout of $G[U]$ can be transformed to another of the form shown in Figure 4.2.1. In Figure 4.2.1, (c'_1, c'_2, c'_3, c'_5) is a permutation of (c_1, c_2, c_3, c_5). Figure 4.2.1 is explicit except that possibly $p_{1,2} = p_{1,6}$, $p_{1,5} = q_{1,6}$, $q_{1,5} = p_{1,6}$, $p_{1,5} = p_{1,4}$, $p_{1,4} = q_{4,5}$, or $p_{4,5} = p_{3,4}$. For convenience, let $c'_4 = c_4$ and $c'_6 = c_6$.

Theorem 4.7 Assume that Figure 4.2.1 is explicit and that $G[U]$ has a correct layout of the form in Figure 4.2.1. Let $S = \{(1,2), (2,3), (3,4), (4,1), (1,5), (1,6), (2,6), (3,5), (4,5), (5,6)\}$ and $T = \{(1,2,6), (1,5,6), (1,4,5), (3,4,5)\}$. Then, for every permutation $\pi = (c'_1, c'_2, c'_3, c'_5)$ of (c_1, c_2, c_3, c_5), the layout in Figure 4.2.1 is a correct layout of C iff the family $\mathcal{F} = \{\mathcal{C}^G_{U,F}(\{c'_i, c'_j\}) \mid (i,j) \in S\} \cup \{\mathcal{C}^G_{U,F}(\{c'_i, c'_j, c'_k\}) \mid (i,j,k) \in T\}$ is a partition of $V(G) - U$.

Proof. Let $\mathcal{E}_{\pi,\max}$ be a maximum extension of the layout of $G[U]$ in Figure 4.2.1. Similarly to Lemma 4.2, we can prove the following:

Lemma 4.8 For every pair $(i,j) \in S$, $\mathcal{C}^G_{U,F}(\{c'_i, c'_j\})$ is exactly the set of countries of $G - U$ embedded in $H[c'_i, c'_j]$ by $\mathcal{E}_{\pi,\max}$. Consequently, for each triple $(i,j,k) \in T$, $\mathcal{C}^G_{U,F}(\{c'_i, c'_j, c'_k\})$ is exactly the set of countries of $G - U$ embedded in $A_{i,j,k}$ by $\mathcal{E}_{\pi,\max}$.

By Lemma 4.8, it is easy to see that the family $\mathcal{F}$ is a partition of $V(G) - U$. To show Theorem 4.7, it suffices to show that whenever $G[U]$ has a correct layout $\mathcal{L}$ of the form in Figure 4.2.1, $\mathcal{E}_{\pi,\max}$ must be a correct layout of $G[U]$, too. We may assume that $\mathcal{L} = \mathcal{E}_{\pi',\max}$ for some permutation π' of (c_1, c_2, c_3, c_5). It remains to show that no matter what π' is, $\mathcal{E}_{\pi',\max}$ can be transformed to $\mathcal{E}_{\pi,\max}$ without altering the correctness. For example, consider the permutation (c_5', c_2', c_3', c_1'). Since $\mathcal{F}$ is a partition of $V(G) - U$, $\mathcal{C}_{U,F}^G(\{c_5', c_2'\}) = \emptyset$, $\mathcal{C}_{U,F}^G(\{c_5', c_2', c_6\}) = \emptyset$, $\mathcal{C}_{U,F}^G(\{c_1', c_3'\}) = \emptyset$, and $\mathcal{C}_{U,F}^G(\{c_1', c_3', c_4'\}) = \emptyset$. Hence, we can assume that $\mathcal{L}$ is of the form in Figure 4.2.2 which is explicit. By Figure 4.2.2, it is clear that $\mathcal{E}_{\pi',\max}$ can be transformed to a correct embedding of G which is an extension of the layout in Figure 4.2.1. ∎

Next, we show how to find a correct layout of C using Theorem 4.7. A useful property is that c_4 must be included in every MC_4 that is 3-sharing with C because C_1 cannot be 4-sharing with two MC_5's. We distinguish several cases as follows.

Case 1: G has no MC_4 that is 3-sharing with C or C_1. Then, we can assume that Figure 4.2.1 is explicit. Thus, by Theorem 4.7, a correct layout of C can be found in polynomial time.

Case 2: G has two MC_4's C_2 and C_3 that are both 3-sharing with C. Then, by Figure 4.2.1, we can assume that $C_2 \cap C = \{c_4, c_1', c_5'\}$ and $C_3 \cap C = \{c_4, c_3', c_5'\}$. Thus, $\langle c_4, c_i, c_j, c_k, c_4 \rangle$ is a correct center of C, where $\{c_i\} = (C_3 \cap C) - C_2$, $\{c_j\} = C - (C_2 \cup C_3)$, and $\{c_k\} = (C_2 \cap C) - C_3$.

Case 3: G has two MC_4's C_2 and C_3 such that C_2 is 3-sharing with C, C_3 is 3-sharing with C_1, and $|C_2 \cap C_3 \cap C| = 0$. By Figure 4.2.1, $C_2 \cap C = \{c_4, c_3', c_5'\}$ and $C_3 \cap C_1 = \{c_1', c_2', c_6\}$. W.l.o.g., let $C_2 = \{c_4, c_3, c_5, c_7\}$ and $C_3 = \{c_6, c_1, c_2, c_8\}$. Let $U_8 = \{c_1, \ldots, c_8\}$. By case 2, we can assume that G has no MC_4 other than C_2 and C_3 that is 3-sharing with C or C_1. Thus, every correct layout of $G[U_8]$ can be transformed to another of one of the four forms in Figure 4.2.3. In each of the forms, (c_1', c_2') is a permutation of (c_1, c_2) and (c_3', c_5') is a permutation of (c_3, c_5). Figure 4.2.3(a) is explicit. Figure 4.2.3(b) becomes explicit after c_7 is removed from it. Figure 4.2.3(c) becomes explicit after c_8 is removed from it. Figure 4.2.3(d) becomes explicit after c_7 and c_8 are removed from it.

Let K_7 be the connected component containing c_7 in the graph $G_7 = G - \{c_3, c_4, c_5\} - \{\{d_1, d_2\} \mid \{d_1, d_2, c_i, c_j\}$ is an MC_4 in G for some $\{c_i, c_j\} \subset \{c_3, c_4, c_5\}\}$. Similarly, let K_8 be the connected component containing c_8 in the graph $G_8 = G - \{c_1, c_2, c_6\} - \{\{d_1, d_2\} \mid \{d_1, d_2, c_i, c_j\}$ is an MC_4 in G for some $\{c_i, c_j\} \subset \{c_1, c_2, c_6\}\}$. The following two lemmas helps us distinguish the four forms.

Lemma 4.9 Suppose that the layout of $G[U_8]$ in Figure 4.2.3(a) (or Figure 4.2.3(b), respectively) is correct. Then, the layout of $G[U_8]$ in Figure 4.2.3(c) (respectively, Figure 4.2.3(d)) is also correct iff $K_8 \cap \{c_3, c_5\} = \emptyset$.

Proof. The necessary condition is obvious. It remains to prove the sufficient condition. Assume that $K_8 \cap \{c_3, c_5\} = \emptyset$. Then, $G[U_8 \cup K_8]$ has a correct layout of the form in Figure 4.2.4. Figure 4.2.4 is explicit except that possibly $r_1 = p_{1,8}$ and $r_2 = p_{6,8}$. Let $\mathcal{E}$ be a correct embedding of G that is an extension of the layout in Figure 4.2.4. Imagine that we start at r_2 and traverse clockwise along the boundary of K_8 in $\mathcal{E}$ until reaching r_1. If we meet no country other than c_6 and c_1' during the traversing, then clearly we can transform the layout in Figure 4.2.3(a) to that in Figure 4.2.3(c). Otherwise, let d be an arbitrary country met by us during the traversing. Note that $d \notin \{c_5, c_3\}$. Since d touches K_8 but is not contained in K_8, there is some country e in K_8 such that $\{c_1', c_6, d, e\}$ is an MC_4 in G and there is no country $f \notin \{c_1', c_6\}$ such that f, d, and e meet each other at the same point in $\mathcal{E}$. So, by Figure 4.2.4, d must be unique and we may assume that there is exactly one point t where d and e meet each other. Now, it should be clear that we can modify $\mathcal{E}$ so that c'_1 and c_6 also meet at t without altering its correctness. Thus, the layout in Figure 4.2.3(a) can be transformed to the layout in Figure 4.2.3(c). ∎

Lemma 4.10 Suppose that the layout of $G[U_8]$ in Figure 4.2.3(a) (or Figure 4.2.3(c), respectively) is correct. Then, the layout of $G[U_8]$ in Figure 4.2.3(b) (respectively, Figure 4.2.3(d)) is also correct iff $K_7 \cap \{c_1, c_2\} = \emptyset$.

Case 3.1: $K_8 \cap \{c_3, c_5\} = K_7 \cap \{c_1, c_2\} = \emptyset$. Then, by Lemma 4.9 and Lemma 4.10, $G[U_8]$ has a correct layout of the form in Figure 4.2.3(d). Since Figure 4.2.3(d) becomes explicit after c_7 and c_8 are removed from it, we can use Theorem 4.7 to compute a correct layout of $G[U]$ in polynomial time.

Case 3.2: $K_8 \cap \{c_3, c_5\} = \emptyset$ but $K_7 \cap \{c_1, c_2\} \neq \emptyset$. Then, by Lemma 4.9 and Lemma 4.10, $G[U_8]$ has a correct layout of the form in Figure 4.2.3(c). In turn, by Figure 4.2.3(c), $dist_{G_7}(c_7, c'_1) < dist_{G_7}(c_7, c'_2)$. Thus, $dist_{G_7}(c_7, c_1) \neq dist_{G_7}(c_7, c_2)$. Suppose that $dist_{G_7}(c_7, c_1) < dist_{G_7}(c_7, c_2)$; the other case is similar. Then, $c'_1 = c_1$ and $c'_2 = c_2$. Let $U_7 = \{c_1, \ldots, c_7\}$ and F_7 be the set of the edges $\{d_1, d_2\}$ in $G - U_7$ such that $\{d_1, d_2, c_i, c_j\}$ is an MC$_4$ in G for some $\{c_i, c_j\} \subset U_7$. Then, for every permutation (c'_3, c'_5), the layout of $G[U_7]$ obtained from Figure 4.2.3(c) by removing c_8 is correct iff $\mathcal{C}^G_{U_7,F_7}(W) = \emptyset$ for all $W \subseteq U_7$ such that $\{c_4, c'_5\} = W$, $\{c'_5, c_2\} \subseteq W$, $\{c'_3, c_1\} \subseteq W$, $\{c'_3, c_6\} \subseteq W$, or $\{c'_3, c_7\} \subseteq W$. This can be shown by by considering a maximum extension of the layout of $G[U_7]$ obtained from Figure 4.2.3(c) by removing c_8.

Case 3.3: $K_7 \cap \{c_1, c_2\} = \emptyset$ but $K_8 \cap \{c_3, c_5\} \neq \emptyset$. This case is similar to Case 3.2.

Case 3.4: $K_7 \cap \{c_1, c_2\} \neq \emptyset$ and $K_8 \cap \{c_3, c_5\} \neq \emptyset$. Then, by Lemma 4.9 and Lemma 4.10, $G[U_8]$ has a correct layout of the form in Figure 4.2.3(a). In turn, by Figure 4.2.3(a), $dist_{G_7}(c_7, c'_1) < dist_{G_7}(c_7, c'_2)$ and $dist_{G_7}(c_8, c'_5) < dist_{G_8}(c_8, c'_3)$. Thus, $dist_{G_7}(c_7, c_1) \neq dist_{G_7}(c_7, c_2)$ and $dist_{G_8}(c_8, c_5) \neq dist_{G_8}(c_8, c_3)$. Suppose that $dist_{G_7}(c_7, c_1) < dist_{G_7}(c_7, c_2)$ and $dist_{G_8}(c_8, c_5) < dist_{G_8}(c_8, c_3)$; other cases are similar. Then, $(c'_1, c'_2, c'_3, c'_5) = (c_1, c_2, c_3, c_5)$.

Case 4: G has two MC$_4$'s C_2 and C_3 such that C_2 is 3-sharing with C, C_3 is 3-sharing with C_1, and $|C_2 \cap C_3 \cap C| = 2$. By Figure 4.2.1, $C_2 \cap C = \{c_4, c'_1, c'_5\}$ and $C_3 \cap C_1 = \{c'_1, c'_5, c_6\}$. W.l.o.g., let $C_2 = \{c_4, c_1, c_5, c_7\}$ and $C_3 = \{c_6, c_1, c_5, c_8\}$.

Case 4.1: $c_7 \neq c_8$ and $\{c_7, c_8\} \notin E(G)$. Let $U_8 = \{c_1, \ldots, c_8\}$. By case 2, we can assume that G has no MC$_4$ other than C_2 and C_3 that is 3-sharing with C or C_1. Thus, every correct layout of $G[U_8]$ can be transformed to another of one of the six forms in Figure 4.2.5. In each of the forms, (c'_1, c'_5) is a permutation of (c_1, c_5) and (c'_2, c'_3) is a permutation of (c_2, c_3). Figure 4.2.5(a) is explicit. Figure 4.2.5(b) and Figure 4.2.5(d) become explicit after c_7 is removed from them. Figure 4.2.5(c) and Figure 4.2.5(e) become explicit after c_8 is removed from them. Figure 4.2.5(f) becomes explicit after c_7 and c_8 are removed from it.

Let K_7 be the connected component containing c_7 in the graph $G_7 = G - \{c_1, c_4, c_5\} - \{\{d_1, d_2\} \mid \{d_1, d_2, c_i, c_j\}$ is an MC$_4$ in G for some $\{c_i, c_j\} \subset \{c_1, c_4, c_5\}\}$. Similarly, let K_8 be the connected component containing c_8 in the graph $G_8 = G - \{c_1, c_5, c_6\} - \{\{d_1, d_2\} \mid \{d_1, d_2, c_i, c_j\}$ is an MC$_4$ in G for some $\{c_i, c_j\} \subset \{c_1, c_5, c_6\}\}$.

Case 4.1.1: $K_7 \cap \{c_2, c_3, c_6\} = \emptyset$ and $K_8 \cap \{c_2, c_3, c_4\} = \emptyset$. Then, similarly to Lemma 4.9, we can prove that $G[U_8]$ has a correct layout of the form in Figure 4.2.5(f). Since Figure 4.2.5(f) becomes explicit after c_7 and c_8 are removed from it, we can use Theorem 4.7 to compute a correct layout of $G[U]$.

Case 4.1.2: $K_8 \cap \{c_2, c_3, c_4\} = \emptyset$ and $K_7 \cap \{c_2, c_3, c_6\} \neq \emptyset$. Then, similarly to Lemma 4.9, we can prove that $G[U_8]$ has a correct layout of the form in Figure 4.2.5(c) or the form in Figure 4.2.5(e). Let $U_7 = \{c_1, \ldots, c_7\}$, and F_7 be the set of the edges $\{d_1, d_2\}$ in $G - U_7$ such that $\{d_1, d_2, c_i, c_j\}$ is an MC_4 in G for some $\{c_i, c_j\} \subset U_7$. By the two figures, one of the following two cases must occur:

Case 4.1.2.1: $dist_{G_7}(c_7, c_6) > \min\{dist_{G_7}(c_7, c_2), dist_{G_7}(c_7, c_3)\}$. Then, $G[U_8]$ has a correct layout of the form in Figure 4.2.5(e). In turn, by Figure 4.2.5(e), $dist_{G_7}(c_7, c'_3) < dist_{G_7}(c_7, c'_2)$. Thus, $dist_{G_7}(c_7, c_3) \neq dist_{G_7}(c_7, c_2)$. Suppose that $dist_{G_7}(c_7, c_3) < dist_{G_7}(c_7, c_2)$; the other case is similar. Then, $c'_3 = c_3$ and $c'_2 = c_2$. Moreover, for every permutation (c'_1, c'_5) of (c_1, c_5), the layout of $G[U_7]$ obtained from Figure 4.2.5(e) by removing c_8 is correct iff $\mathcal{C}^G_{U_7,F_7}(W) = \emptyset$ for all $W \subseteq U_7$ such that $\{c'_5, c_4\} = W$, $\{c'_5, c_2\} \subseteq W$, $\{c'_1, c_3\} \subseteq W$, or $\{c'_1, c_7\} \subseteq W$. This can be shown by considering a maximum extension of the layout of $G[U_7]$ obtained from Figure 4.2.5(e) by removing c_8.

Case 4.1.2.2: $dist_{G_7}(c_7, c_6) < \min\{dist_{G_7}(c_7, c_2), dist_{G_7}(c_7, c_3)\}$. Then, $G[U_8]$ has a correct layout of the form in Figure 4.2.5(c). Moreover, for every permutation (c'_1, c'_5) of (c_1, c_5) and every permutation (c'_2, c'_3) of (c_2, c_3), the layout of $G[U_7]$ obtained from Figure 4.2.5(c) by removing c_8 is correct iff $\mathcal{C}^G_{U_7,F_7}(W) = \emptyset$ for all $W \subseteq U_7$ such that $\{c'_2, c_4\} \subseteq W$, $\{c'_2, c'_5\} \subseteq W$, $\{c'_3, c_6\} \subseteq W$, or $\{c'_3, c'_1\} \subseteq W$. This can be shown by considering a maximum extension of the layout of $G[U_7]$ obtained from Figure 4.2.5(c) by removing c_8.

Case 4.1.3: $K_7 \cap \{c_2, c_3, c_6\} = \emptyset$ and $K_8 \cap \{c_2, c_3, c_4\} \neq \emptyset$. Then, similarly to Lemma 4.9, we can prove that $G[U_8]$ has a correct layout of the form in Figure 4.2.5(d) or the form in Figure 4.2.5(b). Let $U'_8 = \{c_1, \ldots, c_6, c_8\}$, and F'_8 be the set of the edges $\{d_1, d_2\}$ in $G - U'_8$ such that $\{d_1, d_2, c_i, c_j\}$ is an MC_4 in G for some $\{c_i, c_j\} \subset U'_8$. By the two figures, one of the following two cases must occur:

Case 4.1.3.1: $dist_{G_8}(c_8, c_4) > \min\{dist_{G_8}(c_8, c_2), dist_{G_8}(c_8, c_3)\}$. Then, $G[U_8]$ has a correct layout of the form in Figure 4.2.5(d). In turn, by Figure 4.2.5(d), $dist_{G_8}(c_8, c'_2) < dist_{G_8}(c_8, c'_3)$. Thus, $dist_{G_8}(c_8, c_2) \neq dist_{G_8}(c_8, c_3)$. Suppose that $dist_{G_8}(c_8, c_2) < dist_{G_8}(c_8, c_3)$; the other case is similar. Then, $c'_2 = c_2$ and $c'_3 = c_3$. Moreover, for every permutation (c'_1, c'_5) of (c_1, c_5), the layout of $G[U'_8]$ obtained from Figure 4.2.5(d) by removing c_7 is correct iff $\mathcal{C}^G_{U'_8,F'_8}(W) = \emptyset$ for all $W \subseteq U'_8$ such that $\{c'_5, c_8\} \subseteq W$, $\{c'_5, c_2\} \subseteq W$, or $\{c'_1, c_3\} \subseteq W$. This can be shown by considering a maximum extension of the layout of $G[U'_8]$ obtained from Figure 4.2.5(d) by removing c_7.

Case 4.1.3.2: $dist_{G_8}(c_8, c_4) < \min\{dist_{G_8}(c_8, c_2), dist_{G_8}(c_8, c_3)\}$. Then, $G[U_8]$ has a correct layout of the form in Figure 4.2.5(b). Then, for every permutation (c'_1, c'_5) of (c_1, c_5) and every permutation (c'_2, c'_3) of (c_2, c_3), the layout of $G[U'_8]$ obtained from Figure 4.2.5(b) by removing c_7 is correct iff $\mathcal{C}^G_{U'_8,F'_8}(W) = \emptyset$ for all $W \subseteq U'_8$ such that $\{c'_2, c_4\} \subseteq W$, $\{c'_2, c'_5\} \subseteq W$, $\{c'_3, c_6\} \subseteq W$, or $\{c'_3, c'_1\} \subseteq W$. This can be shown by considering a maximum extension of the layout of $G[U'_8]$ obtained from Figure 4.2.5(b) by removing c_7.

Case 4.1.4: $K_7 \cap \{c_2, c_3, c_6\} \neq \emptyset$ and $K_8 \cap \{c_2, c_3, c_4\} \neq \emptyset$. Then, similarly to Lemma 4.9, we can prove that $G[U_8]$ has a correct layout of the form in Figure 4.2.5(a). In turn, by Figure 4.2.5(a), $dist_{G_7}(c_7, c'_3) < dist_{G_7}(c_7, c'_2)$ and $dist_{G_8}(c_8, c'_2) < dist_{G_8}(c_8, c'_3)$. So, we may assume that $dist_{G_7}(c_7, c_3) < dist_{G_7}(c_7, c_2)$ and $dist_{G_8}(c_8, c_2) < dist_{G_8}(c_8, c_3)$; the other case is similar. Then, $c'_2 = c_2$ and $c'_3 = c_3$. Let F_8 be the set of the edges $\{d_1, d_2\}$ in $G - U_8$ such that $\{d_1, d_2, c_i, c_j\}$ is an MC_4 in G. for some $\{c_i, c_j\} \subset U_8$. Then, for every permutation (c_1', c_5') of (c_1, c_5), the layout of $G[U_8]$ in Figure 4.2.5(a) is correct iff $\mathcal{C}^G_{U_8,F_8}(W) = \emptyset$ for all $W \subseteq U_8$ such that $\{c_5', c_2\} \subseteq W$, $\{c_5', c_8\} \subseteq W$, $\{c_1', c_3\} \subseteq W$, or $\{c_1', c_4\} \subseteq W$. This can be shown by considering a maximum extension of the layout of $G[U_8]$ in Figure 4.2.5(a).

Case 4.2: $c_7 \neq c_8$ and $\{c_7, c_8\} \in E(G)$. Let $U_8 = \{c_1, \ldots, c_8\}$. By case 2, we can assume that G has no MC_4 other than C_2 and C_3 that is 3-sharing with C or C_1. Thus, every correct layout of $G[U_8]$ can be transformed to another of one of the three forms in Figure 4.2.6. In each of the forms, (c_1', c_5') is a permutation of (c_1, c_5) and (c_2', c_3') is a permutation of (c_2, c_3). Figure 4.2.6(a) becomes explicit after c_7 is removed from it. Figure 4.2.6(b) becomes explicit after c_8 is removed from it. Figure 4.2.6(c) becomes explicit after c_7 and c_8 are removed from it. Similarly to Case 4.1, we can compute a correct layout of $G[C]$.

Case 4.3: $c_7 = c_8$. Let $U_7 = \{c_1, \ldots, c_7\}$. By case 2, we can assume that G has no MC_4 other than C_2 and C_3 that is 3-sharing with C or C_1. Thus, every correct layout of $G[U_7]$ can be transformed to another of one of the two forms in Figure 4.2.7. In each of the forms, (c_1', c_5') is a permutation of (c_1, c_5) and (c_2', c_3') is a permutation of (c_2, c_3). Figure 4.2.7(a) and Figure 4.2.7(b) become explicit after c_7 is removed from them. To distinguish the two forms in Figure 4.2.7, it suffices to check whether $\mathcal{C}^G_{U_7}(\{c_4, c_7\} \cup W) \neq \emptyset$ for some $W \subset U_7$. If such a W exists, then only the form in Figure 4.2.7(a) is correct; otherwise, the form in Figure 4.2.7(a) can be transformed to the form in Figure 4.2.7(b) and hence the latter is correct. We assume that the form in Figure 4.2.7(a) is correct; the other case is similar. Then, for every permutation (c_1', c_5') of (c_1, c_5) and every permutation (c_2', c_3') of (c_2, c_3), the layout of $G[U]$ obtained from Figure 4.2.7(a) by removing c_7 is correct iff $\mathcal{C}^G_{U,F}(W) = \emptyset$ for all $W \subseteq U$ such that $\{c_3', c_1'\} \subseteq W$, $\{c_3', c_6\} \subseteq W$, $\{c_2', c_5'\} \subseteq W$, or $\{c_2', c_4\} \subseteq W$. This can be shown by considering a maximum extension of the layout of $G[U]$ obtained from Figure 4.2.7(a) by removing c_7.

Case 5: G has two MC_4's C_2 and C_3 such that C_2 is 3-sharing with C, C_3 is 3-sharing with C_1, and $|C_2 \cap C_3 \cap C| = 1$. Let $C_2 \cap C_3 \cap C = \{c_5\}$. By Figure 4.2.1, c_4 and c_6 must be included in C_2 and C_3, respectively. Thus, we may assume that $C_2 = \{c_5, c_4, c_3, c_7\}$ and $C_3 = \{c_5, c_6, c_1, c_8\}$. Let $U_8 = \{c_1, \ldots, c_8\}$.

Case 5.1 $\{c_7, c_8\} \notin E(G)$. By case 2, we can assume that G has no MC_4 other than C_2 and C_3 that is 3-sharing with C or C_1. Thus, every correct layout of $G[U_8]$ can be transformed to another of one of the twelve forms in Figure 4.2.8. Figure 4.2.8(a), Figure 4.2.8(b), Figure 4.2.8(g), and Figure 4.2.8(h) are explicit. Figure 4.2.8(c), Figure 4.2.8(d), and Figure 4.2.8(k) become explicit after c_7 is removed from them. Figure 4.2.8(i), Figure 4.2.8(j), and Figure 4.2.8(e) become explicit after c_8 is removed from them. Figure 4.2.8(f) and Figure 4.2.8(l) are explicit after c_7 and c_8 are removed from them.

Let K_7 be the connected component containing c_7 in the graph $G_7 = G - \{c_3, c_4, c_5\} - \{\{d_1, d_2\} \mid \{d_1, d_2, c_i, c_j\}$ is an MC_4 in G for some $\{c_i, c_j\} \subset \{c_3, c_4, c_5\}\}$. Similarly, let K_8 be the connected component containing c_8 in the graph $G_8 = G - \{c_1, c_5, c_6\} - \{\{d_1, d_2\} \mid \{d_1, d_2, c_i, c_j\}$ is an MC_4 in G for some $\{c_i, c_j\} \subset \{c_1, c_5, c_6\}\}$.

Case 5.1.1: $K_7 \cap \{c_1, c_2, c_6\} = K_8 \cap \{c_2, c_3, c_4\} = \emptyset$. Then, similarly to Lemma 4.9, we can prove that $G[U]$ has a correc layout obtained from Figure 4.2.8(f) or Figure 4.2.8(l) by removing c_7 and c_8. Since Figure 4.2.8(f) and Figure 4.2.8(l) become explicit after c_7 and c_8 are removed from them, the assumption in Theorem 4.7 holds and we can compute a correct layout of $G[U]$.

Case 5.1.2: $K_8 \cap \{c_2, c_3, c_4\} = \emptyset$ and $K_7 \cap \{c_1, c_2, c_6\} \neq \emptyset$. Then, similarly to Lemma 4.9, we can prove that at least one of the layouts of $G[U_8]$ in Figure 4.2.8(e), Figure 4.2.8(i), and Figure 4.2.8(j) is correct. By the three layouts, one of the following three cases must occur:

Case 5.1.2.1: $dist_{G_7}(c_7, c_1) < \min\{dist_{G_7}(c_7, c_2), dist_{G_7}(c_7, c_6)\}$. Then, the layout of $G[U_8]$ in Figure 4.2.8(e) is correct.

Case 5.1.2.2: $dist_{G_7}(c_7, c_2) < \min\{dist_{G_7}(c_7, c_1), dist_{G_7}(c_7, c_6)\}$. Then, the layout of $G[U_8]$ in Figure 4.2.8(j) is correct.

Case 5.1.2.3: $dist_{G_7}(c_7, c_6) < \min\{dist_{G_7}(c_7, c_1), dist_{G_7}(c_7, c_2)\}$. Then, the layout of $G[U_8]$ in Figure 4.2.8(i) is correct.

Case 5.1.3: $K_8 \cap \{c_2, c_3, c_4\} \neq \emptyset$ and $K_7 \cap \{c_1, c_2, c_6\} = \emptyset$. Then, similarly to Lemma 4.9, we can prove that at least one of the layouts of $G[U_8]$ in Figure 4.2.8(c), Figure 4.2.8(d), and Figure 4.2.8(k) is correct. By the three layouts, one of the following three cases must occur:

Case 5.1.3.1: $dist_{G_8}(c_8, c_4) < \min\{dist_{G_8}(c_8, c_2), dist_{G_8}(c_8, c_3)\}$. Then, the layout of $G[U_8]$ in Figure 4.2.8(c) is correct.

Case 5.1.3.2: $dist_{G_8}(c_8, c_2) < \min\{dist_{G_8}(c_8, c_3), dist_{G_8}(c_8, c_4)\}$. Then, the layout of $G[U_8]$ in Figure 4.2.8(d) is correct.

Case 5.1.3.3: $dist_{G_8}(c_8, c_3) < \min\{dist_{G_8}(c_8, c_2), dist_{G_8}(c_8, c_4)\}$. Then, the layout of $G[U_8]$ in Figure 4.2.8(k) is correct.

Case 5.1.4: $K_7 \cap \{c_1, c_2, c_6\} \neq \emptyset$ and $K_8 \cap \{c_2, c_3, c_4\} \neq \emptyset$. Then, similarly to Lemma 4.9, we can prove that at least one of the four layouts of $G[U_8]$ in Figure 4.2.8(a), Figure 4.2.8(b), Figure 4.2.8(g), and Figure 4.2.8(h) is correct. By the four layouts, if c_7 and c_8 belong to the same connected components in the graph $G - U$, then at least one of the two layouts of $G[U_8]$ in Figure 4.2.8(b) and Figure 4.2.8(h) is correct; otherwise, at least one of the two layouts of $G[U_8]$ in Figure 4.2.8(a) and Figure 4.2.8(g) is correct.

Case 5.1.4.1: There are two distinct connected components K'_7 and K'_8 in $G - U$ containing c_7 and c_8, respectively. Then, at least one of the layouts of $G[U_8]$ in Figure 4.2.8(a) and Figure 4.2.8(g) is correct. If the layout in Figuer 4.2.8(a) is correct, then $N_G(K'_7) \cap \{c_1, c_2, c_6\} = \{c_1\}$ and $N_G(K'_8) \cap \{c_2, c_3, c_4\} = \{c_2\}$. On the other hand, if the layout in Figuer 4.2.8(g) is correct, then $N_G(K'_7) \cap \{c_1, c_2, c_6\} = \{c_2\}$ and $N_G(K'_8) \cap \{c_2, c_3, c_4\} = \{c_3\}$. Thus, we can tell which one of the two layouts is correct.

Case 5.1.4.2: There is a connected component K' in $G - U$ containing both c_7 and c_8. Then, at least one of the layouts of $G[U_8]$ in Figure 4.2.8(b) and Figure 4.2.8(h) is correct. If the layout in Figuer 4.2.8(b) is correct, then $\{c_3, c_6\} \cap N_G(K' - \{c_7, c_8\}) = \emptyset$. On the other hand, if the layout in Figuer 4.2.8(h) is correct, then $\{c_1, c_4\} \cap N_G(K' - \{c_7, c_8\}) = \emptyset$. Thus, we may assume that $\{c_1, c_3, c_4, c_6\} \cap N_G(K' - \{c_7, c_8\}) = \emptyset$; otherwise, we can tell which one of the two layouts is correct. Let $\mathcal{W}_b$ be the family of the sets $\{c_1, c_2\}$, $\{c_2, c_3\}$, $\{c_3, c_4\}$, $\{c_4, c_1\}$, $\{c_1, c_6\}$, $\{c_2, c_6\}$, $\{c_1, c_2, c_6\}$, $\{c_5, c_6\}$, and $\{c_3, c_5\}$. Let $\mathcal{W}_h$ be the family of the sets $\{c_1, c_2\}$, $\{c_2, c_4\}$, $\{c_4, c_5\}$, $\{c_5, c_1\}$, $\{c_1, c_6\}$, $\{c_2, c_3\}$, $\{c_2, c_3, c_4\}$, $\{c_3, c_4\}$, and $\{c_3, c_6\}$. Then, by considering the maximum extensions of the two layouts, it is easy to see that (1) the layout of $G[U_8]$ in Figure 4.2.8(b) is correct iff the family $\{\mathcal{C}_U^G(W) \mid W \in \mathcal{W}_b\}$ is a partition of $V(G) - (U \cup K')$, and (2) the layout of $G[U_8]$ in Figure 4.2.8(h) is correct iff the family $\{\mathcal{C}_U^G(W) \mid W \in \mathcal{W}_h\}$ is a partition of $V(G) - (U \cup K')$.

Case 5.2 $\{c_7, c_8\} \in E(G)$. By case 2, we can assume that G has no MC_4 other than C_2 and C_3 that is 3-sharing with C or C_1. Thus, every correct layout of $G[U_8]$ can be transformed to another of one of the twelve forms in Figure 4.2.9. Figure 4.2.9(a) and Figure 4.2.9(b) become explicit after c_7 is removed from them. Similarly to Case 5.1.4.2, we can tell which of the two layouts in Figure 4.2.9 is correct.

Case 6: G has exactly one MC_4 C_2 that is 3-sharing with C but has no MC_4 that is 3-sharing with C_1. Since c_4 must be included in C_2, we may let $C_2 = \{c_1, c_4, c_5, c_7\}$. Let $U_7 = \{c_1, \ldots, c_7\}$. Then by Figure 4.2.10, every correct layout of $G[U_7]$ can be transformed to another of one of the five forms in Figure 4.2.10. In all the forms, (c'_1, c'_5) is a permutation of (c_1, c_5) and (c'_2, c'_3) is a permutation of (c_2, c_3). Figure 4.2.10(a), Figure 4.2.10(b), and Figure 4.2.10(d) are explicit. Figure 4.2.10(c) and Figure 4.2.10(e) become explicit after c_7 is removed from them.

Let K_7 be the connected component containing c_7 in the graph $G_7 = G - \{c_1, c_4, c_5\} - \{\{d_1, d_2\} \mid \{d_1, d_2, c_i, c_j\}$ is an MC_4 in G for some $\{c_i, c_j\} \subset \{c_1, c_4, c_5\}\}$.

Case 6.1: $K_7 \cap \{c_2, c_3, c_6\} = \emptyset$. Then, similarly to Lemma 4.9, we can prove that $G[U_7]$ has a correct layout of the form in Figure 4.2.10(c) or the form in Figure 4.2.10(e). Thus, the assumption in Theorem 4.7 holds and we can compute a correct layout of $G[U]$.

Case 6.2: $K_7 \cap \{c_2, c_3, c_6\} \neq \emptyset$. Then, similarly to Lemma 4.9, we can prove that $G[U_7]$ has a correct layout of the form in Figure 4.2.10(a), the form in Figure 4.2.10(b), or the form in Figure 4.2.10(d). By the three forms, one of the following two cases must occur:

Case 6.2.1: $dist_{G_7}(c_7, c_6) < \min\{dist_{G_7}(c_7, c_2), dist_{G_7}(c_7, c_3)\}$. Then, $G[U_7]$ has a correct layout of the form in Figure 4.2.10(b). Similarly to Case 4.1.2.2, we can compute a correct layout of $G[U]$.

Case 6.2.2: $dist_{G_7}(c_7, c_6) > \min\{dist_{G_7}(c_7, c_2), dist_{G_7}(c_7, c_3)\}$. Then, $G[U_7]$ has a correct layout of the form in Figure 4.2.10(a) or the form in Figure 4.2.10(d). By the two forms, $dist_{G_7}(c_7, c_2) \neq dist_{G_7}(c_7, c_3)$. Moreover, if $dist_{G_7}(c_7, c_3) < dist_{G_7}(c_7, c_2)$, then $c'_3 = c_3$; otherwise, $c'_3 = c_2$. Suppose that $c'_3 = c_3$; the other case is similar. Then, $c'_2 = c_2$. To distinguish the two forms, we check whether there is some connected component K in $G - U_7$ that is adjacent to both c_3 and c_6 in G. If K exists, then only the form in Figure 4.2.10(d) is correct; otherwise, by considering maximum extensions of the two layouts of $G[U]$ obtained from Figure 4.2.10(a) and Figure 4.2.10(d) by removing c_7, we can prove that the form in Figure 4.2.10(d) can be transformed to the form in Figure 4.2.10(a) without altering the correctness. Thus, we know which one of the two forms in Figure 4.2.10(a) and Figure 4.2.10(d) is correct. If it is Figure 4.2.10(a), then similarly to Case 4.1.2.1, we can compute a correct layout of $G[U]$. If it is Figure 4.2.10(d), then similarly to Case 3.2, we can compute a correct layout of $G[U]$.

4.3 The case having no 4-sharing MC_5

By the discussions in § 4.2, we can assume that no MC_5 in G is 4-sharing with C. Then, by Theorem 4.1, we can further assume that at least one MC_4 in G is 3-sharing with C. If there are three or more MC_4's in G that are 3-sharing with C, then by Fact 4, it is easy to figure out a correct layout of C. So, we may assume that either one or two MC_4's in G are 3-sharing with C. Let F be the set of the edges $\{d_1, d_2\}$ in $G - C$ such that $\{d_1, d_2, c_i, c_j\}$ is an MC_4 in G for some $\{c_i, c_j\} \subset C$.

Case 1: Exactly one MC_4 C_1 in G is 3-sharing with C. Let $C_1 = \{c_2, c_3, c_5, c_6\}$. By Figure 4.1, every correct layout of $G[U]$ can be transformed to another of one of the three forms in Figure 4.3.1. In all the forms, (c'_2, c'_3, c'_5) is a permutation of (c_2, c_3, c_5).

Figure 4.3.1(a) and Figure 4.3.1(b) are explicit. Figure 4.3.1(c) becomes explicit after c_6 is removed from it. Let K_6 be the connected component containing c_6 in the graph $G_6 = G - \{c_2, c_3, c_5\} - \{\{d_1, d_2\} \mid \{d_1, d_2, c_i, c_j\}$ is an MC_4 in G for some $\{c_i, c_j\} \subset \{c_2, c_3, c_5\}\}$.

Case 1.1: $\{c_1, c_4\} \cap K_6 = \emptyset$. Then, similarly to Lemma 4.9, we can prove that C has a correct layout of the form in Figure 4.3.1(c). Thus, the assumption in Theorem 4.1 holds and a correct layout of C can be easily computed.

Case 1.2: $\{c_1, c_4\} \cap K_6 \neq \emptyset$. Then, similarly to Lemma 4.9, we can prove that C has a correct layout of the form in Figure 4.3.1(a) or the form in Figure 4.3.1(b). By the two forms, one of the following two cases must occur:

Case 1.2.1: $dist_{G_6}(c_6, c_1) < dist_{G_6}(c_6, c_4)$. Let $U_6 = \{c_1, \ldots, c_6\}$, and F_6 be the set of the edges $\{d_1, d_2\}$ in $G - U_6$ such that $\{d_1, d_2, c_i, c_j\}$ is an MC_4 in G for some $\{c_i, c_j\} \subset U_6$. Then, for every permutation (c'_2, c'_3, c'_5), the layout of $G[U_6]$ in Figure 4.3.1(b) is correct iff $\mathcal{C}^G_{U_6,F_6}(\{c'_2, c'_5\}) = \emptyset$ and $\mathcal{C}^G_{U_6,F_6}(W) = \emptyset$ for all $W \subseteq U_6$ such that $\{c'_2, c_4\} \subseteq W$, $\{c'_3, c_1\} \subseteq W$, or $\{c'_3, c_6\} \subseteq W$. This can be shown by considering a maximum extension of the layout of $G[U_6]$ in Figure 4.3.1(b).

Case 1.2.2: $dist_{G_6}(c_6, c_4) < dist_{G_6}(c_6, c_1)$. This case is similar to Case 1.2.1.

Case 2: Exactly two MC_4's C_1 and C_2 in G are 3-sharing with C and $|C_1 \cap C_2| = 1$. Let $C_1 = \{c_1, c_2, c_5, c_6\}$ and $C_2 = \{c_3, c_4, c_5, c_7\}$. Let $U_7 = \{c_1, \ldots, c_7\}$. Then by Fact 4, only c_5 can be a correct crust of C. Let K_6 be the connected component containing c_6 in the graph $G_6 = G - \{c_1, c_2, c_5\} - \{\{d_1, d_2\} \mid \{d_1, d_2, c_i, c_j\}$ is an MC_4 in G for some $\{c_i, c_j\} \subset \{c_1, c_2, c_5\}\}$. Similarly, let K_7 be the connected component containing c_7 in the graph $G_7 = G - \{c_3, c_4, c_5\} - \{\{d_1, d_2\} \mid \{d_1, d_2, c_i, c_j\}$ is an MC_4 in G for some $\{c_i, c_j\} \subset \{c_3, c_4, c_5\}\}$.

Case 2.1 $\{c_6, c_7\} \notin E(G)$. Then, by Figure 4.1, every correct layout of $G[U_7]$ can be transformed to another of one of the five forms in Figure 4.3.2. In all the forms, (c'_1, c'_2) is a permutation of (c_1, c_2) and (c'_3, c'_4) is a permutation of (c_3, c_4). Figure 4.3.2(a) and Figure 4.3.2(b) are explicit. Figure 4.3.2(c) becomes explicit after c_7 is removed from it. Figure 4.3.2(d) becomes explicit after c_6 is removed from it. Figure 4.3.2(e) becomes explicit after c_6 and c_7 are removed from it.

Case 2.1.1: $\{c_3, c_4\} \cap K_6 = \emptyset$ and $\{c_1, c_2\} \cap K_7 = \emptyset$. Then, similarly to Lemma 4.9, we can prove that C has a correct layout of the form in Figure 4.3.2(e). Thus, the assumption in Theorem 4.1 holds and a correct layout of C can be easily computed.

Case 2.1.2: $\{c_3, c_4\} \cap K_6 = \emptyset$ but $\{c_1, c_2\} \cap K_7 \neq \emptyset$. Then, similarly to Lemma 4.9, we can prove that C has a correct layout of the form in Figure 4.3.2(d). By this figure, $dist_{G_7}(c_7, c_1) \neq dist_{G_7}(c_7, c_2)$. Suppose that $dist_{G_7}(c_7, c_1) < dist_{G_7}(c_7, c_2)$; the other case is similar. Then, $c'_1 = c_1$ and $c'_2 = c_2$. Thus, similarly to Case 1.2.1, we can compute a correct layout of C.

Case 2.1.3: $\{c_1, c_2\} \cap K_7 = \emptyset$ but $\{c_3, c_4\} \cap K_6 \neq \emptyset$. This case is similar to Case 2.1.2.

Case 2.1.4: $\{c_3, c_4\} \cap K_6 \neq \emptyset$ and $\{c_1, c_2\} \cap K_7 \neq \emptyset$. Then, similarly to Lemma 4.9, we can prove that C has a correct layout of the form in Figure 4.3.2(a) or Figure 4.3.2(b).

Case 2.1.4.1: c_6 and c_7 belong to the same connected component in $G - C$. Then, $G[U_7]$ has a correct layout of the form in Figure 4.3.2(b). Let K be the connected component in $G - C$ containing both c_6 and c_7. By Figure 4.3.2(b), $|N_G(K - \{c_6\}) \cap \{c_1, c_2\}| \leq 1$ and $|N_G(K - \{c_7\}) \cap \{c_3, c_4\}| \leq 1$.

Case 2.1.4.1.1: $|N_G(K - \{c_6\}) \cap \{c_1, c_2\}| = |N_G(K - \{c_7\}) \cap \{c_3, c_4\}| = 1$. Suppose that $N_G(K - \{c_6\}) \cap \{c_1, c_2\} = \{c_2\}$ and $N_G(K - \{c_7\}) \cap \{c_3, c_4\} = \{c_3\}$; the other cases are similar. Then, $(c'_1, c'_2) = (c_1, c_2)$ and $(c'_3, c'_4) = (c_3, c_4)$.

Case 2.1.4.1.2: $|N_G(K - \{c_6\}) \cap \{c_1, c_2\}| = 1$ and $|N_G(K - \{c_7\}) \cap \{c_3, c_4\}| = 0$. Suppose that $N_G(K - \{c_6\}) \cap \{c_1, c_2\} = \{c_2\}$; the other case is similar. Then, $(c'_1, c'_2) = (c_1, c_2)$. Moreover, for every permutation (c'_3, c'_4) of (c_3, c_4), the layout of $G[U_7]$ in Figure 4.3.2(b) is correct iff the family $\{\mathcal{C}^G_{U_7}(W) \mid W \in \mathcal{W}\}$ is a partition of $V(G) - (C \cup K)$, where $\mathcal{W}$ is the class of the sets $\{c_1, c_2\}$, $\{c_2, c'_3\}$, $\{c_3, c_4\}$, $\{c'_4, c_1\}$, $\{c_1, c_5\}$, $\{c'_4, c_5\}$, and $\{c_1, c'_4, c_5\}$. This can be shown by considering a maximum extension of the layout of $G[U_7]$ in Figure 4.3.2(b).

Case 2.1.4.1.3: $|N_G(K - \{c_6\}) \cap \{c_1, c_2\}| = 0$ and $|N_G(K - \{c_7\}) \cap \{c_3, c_4\}| = 1$. Similar to Case 2.1.4.1.2.

Case 2.1.4.1.4: $|N_G(K - \{c_6\}) \cap \{c_1, c_2\}| = 0$ and $|N_G(K - \{c_7\}) \cap \{c_3, c_4\}| = 0$. Then, for every permutation (c'_1, c'_2) of (c_1, c_2) and every permutation (c'_3, c'_4) of (c_3, c_4), the layout of $G[U_7]$ in Figure 4.3.2(b) is correct iff the family $\{\mathcal{C}^G_{U_7}(W) \mid W \in \mathcal{W}\}$ is a partition of $V(G) - (C \cup K)$, where $\mathcal{W}$ is the class of the sets $\{c'_1, c'_2\}$, $\{c'_2, c'_3\}$, $\{c'_3, c'_4\}$, $\{c'_4, c'_1\}$, $\{c'_1, c_5\}$, $\{c'_4, c_5\}$, and $\{c'_1, c'_4, c_5\}$. This can be shown by considering a maximum extension of the layout of $G[U_7]$ in Figure 4.3.2(b).

Case 2.1.4.2: c_6 and c_7 do not belong to the same connected component in $G - C$. Then, $G[U_7]$ has a correct layout of the form in Figure 4.3.2(a). By this figure, $dist_{G_6}(c_6, c_3) \neq dist_{G_6}(c_6, c_4)$ and $dist_{G_7}(c_7, c_1) \neq dist_{G_7}(c_7, c_2)$. Suppose that $dist_{G_6}(c_6, c_3) < dist_{G_6}(c_6, c_4)$ and $dist_{G_7}(c_7, c_1) < dist_{G_7}(c_7, c_2)$; the other cases are similar. Then, $(c'_1, c'_2) = (c_1, c_2)$ and $(c'_3, c'_4) = (c_3, c_4)$.

Case 2.2: $\{c_6, c_7\} \in E(G)$. Then, every correct layout of $G[U_7]$ can be transformed to another of the form in Figure 4.3.3. In Figure 4.3.3, (c'_1, c'_2) is a permutation of (c_1, c_2) and (c'_3, c'_4) is a permutation of (c_3, c_4). Figure 4.3.3 becomes explicit after c_7 is removed from it. Similarly to Case 2.1.4.1, we can compute a correct layout of $G[U_7]$.

Case 3: Exactly two MC_4's C_1 and C_2 in G are 3-sharing with C and $|C_1 \cap C_2| = 2$. Let $C_1 = \{c_2, c_3, c_5, c_6\}$ and $C_2 = \{c_3, c_4, c_5, c_7\}$. Let $U_7 = \{c_1, \ldots, c_7\}$.

Case 3.1: $\{c_6, c_7\} \notin E(G)$. Then, every correct layout of $G[U_7]$ can be transformed to another of one of the six forms in Figure 4.3.4. In all the forms, (c'_3, c'_5) is a permutation of (c_3, c_5). Figure 4.3.4(a) is explicit. Figure 4.3.4(b) and Figure 4.3.4(d) become explicit after c_7 is removed from them. Figure 4.3.4(c) and Figure 4.3.4(e) become explicit after c_6 is removed from them. Figure 4.3.4(f) becomes explicit after c_6 and c_7 are removed from it. Let K_6 be the connected component containing c_6 in the graph $G_6 = G - \{c_2, c_3, c_5\} - \{\{d_1, d_2\} \mid \{d_1, d_2, c_i, c_j\}$ is an MC_4 in G for some $\{c_i, c_j\} \subset \{c_2, c_3, c_5\}\}$. Similarly, let K_7 be the connected component containing c_7 in the graph $G_7 = G - \{c_3, c_4, c_5\} - \{\{d_1, d_2\} \mid \{d_1, d_2, c_i, c_j\}$ is an MC_4 in G for some $\{c_i, c_j\} \subset \{c_3, c_4, c_5\}\}$.

Case 3.1.1: $\{c_1, c_4\} \cap K_6 = \emptyset$ and $\{c_1, c_2\} \cap K_7 = \emptyset$. Then, similarly to Lemma 4.9, we can prove that $G[U_7]$ has a correct layout of the form in Figure 4.3.4(f). Thus, the assumption in Theorem 4.1 holds and a correct layout of C can be easily computed.

Case 3.1.2: $\{c_1, c_4\} \cap K_6 = \emptyset$ but $\{c_1, c_2\} \cap K_7 \neq \emptyset$. Then, similarly to Lemma 4.9, we can prove that $G[U_7]$ has a correct layout of the form in Figure 4.3.4(c) or the form in Figure 4.3.4(e). By the two forms, $dist_{G_7}(c_7, c_1) \neq dist_{G_7}(c_7, c_2)$. If $dist_{G_7}(c_7, c_1) > dist_{G_7}(c_7, c_2)$, then $G[U_7]$ has a correct layout of the form in Figure 4.3.4(e); otherwise, $G[U_7]$ has a correct layout of the form in Figure 4.3.4(c). In either case, similarly to Case 1.2.1, we can compute a correct layout of C.

Case 3.1.3: $\{c_1, c_2\} \cap K_7 = \emptyset$ but $\{c_1, c_4\} \cap K_6 \neq \emptyset$. This case is similar to Case 3.1.2.

Case 3.1.4: $\{c_1, c_4\} \cap K_6 \neq \emptyset$ and $\{c_1, c_2\} \cap K_7 \neq \emptyset$. Then, similarly to Lemma 4.9, we can prove that $G[U_7]$ has a correct layout of the form in Figure 4.3.4(a). Let F_7 be the set of the edges $\{d_1, d_2\}$ in $G - U_7$ such that $\{d_1, d_2, c_i, c_j\}$ is an MC_4 in G for some $\{c_i, c_j\} \subset U_7$. Then, for every permutation (c'_3, c'_5), the layout of $G[U_7]$ in Figure 4.3.4(a) is correct iff $\mathcal{C}^G_{U_7,F_7}(\{c'_2, c'_5\}) = \emptyset$, $\mathcal{C}^G_{U_7,F_7}(\{c'_4, c'_5\}) = \emptyset$, and $\mathcal{C}^G_{U_7,F_7}(W) = \emptyset$ for all $W \subseteq U_7$ such that $\{c'_3, c_1\} \subseteq W$, $\{c'_3, c_6\} \subseteq W$, or $\{c'_3, c_7\} \subseteq W$. This can be shown by considering a maximum extension of the layout of $G[U_7]$ in Figure 4.3.4(a).

Case 3.2: $\{c_6, c_7\} \in E(G)$. Then, every correct layout of $G[U_7]$ can be transformed to another of one of the two forms in Figure 4.3.5. In both forms, (c'_3, c'_5) is a permutation of (c_3, c_5). Figure 4.3.5(a) becomes explicit after c_7 is removed from it. Figure 4.3.5(b) becomes explicit after c_6 is removed from it. Similarly to Case 3.1, we can tell which of the two layouts in Figure 4.3.5 is correct and further compute a correct permutation of (c_3, c_5).

5 Removing maximal cliques of size 4

Recall that G is assumed to have a correct embedding. By the arguments in § 3 and § 4, we may further assume that G has neither MC_6 nor MC_5. Let $C = \{c_1, \ldots, c_4\}$ be an MC_4 in G. We try to find a correct 4-point via constructing a correct layout of C. If we fail eventually, we will be able to decompose G into smaller graphs. It is easy to see that every correct layout of C can be transformed to another of one of the forms in Figure 5.1. In each form except the one in Figure 5.1(b), $(c'_1, \ldots, c'_4)$ is a permutation of $(c_1, \ldots, c_4)$. Figure 5.1(a) through Figure 5.1(f) are explicit. Hereafter, a correct layout of C always means one that is of one of the forms in Figure 5.1. A layout of the form in Figure 5.1(a) is called a *pizza*, while a layout of one of the rest four forms is called a *nonpizza*. The layout in Figure 5.1(b) is called the *rice ball*. A layout of the form in Figure 5.1(c) is called a *1-type nonpizza*. A layout of the form in Figure 5.1(d) or Figure 5.1(e) is called a *2-type nonpizza*. A layout of the form in Figure 5.1(f) is called a *3-type nonpizza*. By Figure 5.1, it is easy to see that for every $W \subseteq C$ with $|W| = 3$, G has at most one MC_4 C' other than C with $W \subseteq C'$.

Lemma 5.1 Let C_1 be an MC_4 in G 3-sharing with C. Then, the following (1) and (2) hold:

(1) If C has a correct nonpizza layout in which there is no point where the three countries in $C \cap C_1$ meet each other, then the unique country in $C - C_1$ and the unique country in $C_1 - C$ do not belong to the same connected component in the graph $G' = G - (C \cap C_1) - \{\{d_1, d_2\} \mid \{d_1, d_2, c_i, c_j\}$ is an MC_4 in G for some $\{c_i, c_j\} \subseteq C \cap C_1\}$.

(2) For every correct nonpizza layout $\mathcal{L}$ of C in which there is a point where the three countries in $C \cap C_1$ meet each other, if the unique country in $C - C_1$ and the unique country in $C_1 - C$ do not belong to the same connected component in G', then C has another correct nonpizza layout $\mathcal{L}'$ such that

(2.1) there is no point in $\mathcal{L}'$ where the three countries in $C \cap C_1$ meet each other and

(2.2) for every $U \subseteq C$ such that $|U| = 3$ and $U \neq C \cap C_1$, there is a point in $\mathcal{L}'$ where the three countries in U meet each other iff there is a point in $\mathcal{L}$ where the three countries in U meet each other.

Proof. (1) Let $\mathcal{L}$ be a correct nonpizza layout of C in which there is no point where the three countries in $C \cap C_1$ meet each other. Then, there must exist a hole $\mathcal{H}$ in $\mathcal{L}$ whose boundary is enclosed by the three countries in $C \cap C_1$. Since the country in $C - C_1$ and the country in $C_1 - C$ cannot touch each other and both C and C_1 are a clique, c_5 must be embedded in the hole $\mathcal{H}$ in every correct embedding of G that is an extension of $\mathcal{L}$. From this, it is easy to see that (1) holds.

(2) It suffices to show that if the unique country in $C - C_1$ and the unique country in $C_1 - C$ do not belong to the same connected component in G', then independent of the permutation (c'_1, c'_2, c'_3, c'_4) of (c_1, c_2, c_3, c_4), the layout in Figure 5.1(f) can be transformed to the one in Figure 5.1(e) or the one in Figure 5.1(d), the layouts in Figure 5.1(e) and Figure 5.1(d) can be transformed to the one in Figure 5.1(c), and the layout in Figure 5.1(c) can be transformed to the one in Figure 5.1(b). This can be shown similarly to Lemma 4.9. ∎

Definition 5.2 The *best nonpizza layout* of C is a nonpizza layout of C which is correct whenever C has a correct nonpizza layout, and contains the fewest points where three countries in C meet each other.

For example, when there is no MC_4 in G 3-sharing with C, the rice-ball layout is the best nonpizza layout. By Lemma 5.1, we know how to compute the best nonpizza layout of C.

5.1 Distinguishing pizzas and the rice-ball

Assume that the rice-ball layout is the best nonpizza layout of C. Our goal is to decide whether the rice-ball layout is correct under this assumption. Let F be the set of the edges $\{d_1, d_2\}$ in $G - C$ such that $\{d_1, d_2, c_i, c_j\}$ is an MC_4 in G for some $\{c_i, c_j\} \subset C$. Let $S = \{\{c_i, c_j\} \mid 1 \le i < j \le 4\}$ and $T = \{\{c_i, c_j, c_k\} \mid 1 \le i < j < k \le 4\}$.

Theorem 5.3 The rice-ball layout is correct iff the following (1) through (4) hold:

(1) The family $\mathcal{F} = \{\mathcal{C}^G_{C,F}(W) \mid W \in S\} \cup \{\mathcal{C}^G_{C,F}(W) \mid W \in T\}$ is a partition of $V(G) - C$.

(2) Let $\mathcal{G}$ be the graph $(\mathcal{F}, E')$, where E' consists of the edges $\{X, Y\}$ such that $X \in \mathcal{F}$, $Y \in \mathcal{F}$, and $X \cap N_G(Y) \neq \emptyset$. Then, $\mathcal{G}$ is a subgraph of the graph $\mathcal{G}_a$ in Figure 5.2.

(3) For each $W \in S$ such that $deg_{\mathcal{G}_a}(\mathcal{C}^G_{C,F}(W)) = 2$, the edge between the two neighbors of $\mathcal{C}^G_{C,F}(W)$ in $\mathcal{G}_a$ is also in $\mathcal{G}$ only if $\mathcal{C}^G_{C,F}(W) = \emptyset$.

(4) For each edge $\{X, Y\} = \{\mathcal{C}^G_{C,F}(W_1), \mathcal{C}^G_{C,F}(W_2)\}$ in $\mathcal{G}$, $|N_G(X) \cap Y| = 1$, $|X \cap N_G(Y)| = 1$, and $(W_1 \cap W_2) \cup (N_G(X) \cap Y) \cup (X \cap N_G(Y))$ is an MC_4 in G and is the unique maximal clique in G containing the two countries in $(X \cap N_G(Y)) \cup (N_G(X) \cap Y)$.

Proof. It is easy to prove the necessary condition by considering a maximum extension of the rice-ball layout. Next, we show the sufficient condition. To do this, it suffices to show that whenever (1) through (4) hold, every correct pizza layout of C can be transformed to the rice-ball layout. Assume that (1) through (4) hold and C has a correct pizza layout $\mathcal{L}_p$. We further assume that the four countries in C meet each other at the center of $\mathcal{L}_p$ in the cyclic order c_1, ..., c_4, c_1; other cases are similar. Similarly to Lemma 4.2, we can prove the following claim:

Claim 1 Let $\mathcal{E}_{\max}$ be a maximum extension of $\mathcal{L}_p$. Then, for every $\{c_i, c_j\} \in S - \{\{c_1, c_3\}, \{c_2, c_4\}\}$, $\mathcal{C}^G_{C,F}(W)$ is exactly the set of countries of $G - C$ embedded in $H[c_i, c_j]$ by $\mathcal{E}_{\pi,\max}$.

To see that $\mathcal{L}_p$ can be transformed to the rice-ball layout, it suffices to consider the following five cases:

Case 1: Every connected component in $\mathcal{G}$ contains at most one node $\mathcal{C}^G_{C,F}(W)$ with $W \in T$. Obviously, if there is no $W \in T$ with $\mathcal{C}^G_{C,F}(W) \neq \emptyset$, then $\mathcal{L}_p$ can be transformed to the rice-ball layout. So, we may assume that $\mathcal{C}^G_{C,F}(W) \neq \emptyset$ for some $W \in T$. Suppose that $W = \{c_1, c_2, c_3\}$; other cases are similar. Then, it is clear that we can modify $\mathcal{L}_p$ so that c_1 and c_3 (but no another country) meet each other at a point q other than the center of $\mathcal{L}_p$ without altering the correctness of $\mathcal{L}_p$. Now, it is easy to see that the modified $\mathcal{L}_p$ can be transformed to the rice-ball layout.

Case 2: For all $W \in T$, $\mathcal{C}^G_{C,F}(W) \neq \emptyset$. Then, in $\mathcal{L}_p$, $\mathcal{C}^G_{C,F}(\{c_1, c_2, c_3\})$ has to touch $\mathcal{C}^G_{C,F}(\{c_1, c_2, c_4\})$, $\mathcal{C}^G_{C,F}(\{c_1, c_2, c_4\})$ has to touch $\mathcal{C}^G_{C,F}(\{c_1, c_3, c_4\})$, $\mathcal{C}^G_{C,F}(\{c_1, c_3, c_4\})$ has to touch $\mathcal{C}^G_{C,F}(\{c_2, c_3, c_4\})$, and $\mathcal{C}^G_{C,F}(\{c_2, c_3, c_4\})$ has to touch $\mathcal{C}^G_{C,F}(\{c_1, c_2, c_3\})$. Thus, by (3) and Figure 5.1(a), $\mathcal{C}^G_{C,F}(W) = \emptyset$ for all $W \in S$. Moreover, by (1) through (4), it is easy to see that G has a correct embedding shown in Figure 5.3. ¿From this embedding, it is easy to see that the rice-ball layout is correct.

Case 3: For exactly one $W \in T$, $\mathcal{C}^G_{C,F}(W) = \emptyset$. We assume that $\mathcal{C}^G_{C,F}(\{c_2, c_3, c_4\}) = \emptyset$; other cases are similar. Then, in $\mathcal{L}_p$, $\mathcal{C}^G_{C,F}(\{c_1, c_2, c_3\})$ has to touch $\mathcal{C}^G_{C,F}(\{c_1, c_2, c_4\})$, and $\mathcal{C}^G_{C,F}(\{c_1, c_2, c_4\})$ has to touch $\mathcal{C}^G_{C,F}(\{c_1, c_3, c_4\})$. Thus, by (3) and Figure 5.1(a), $\mathcal{C}^G_{C,F}(W) = \emptyset$ for all $W \in S - \{\{c_2, c_3\}, \{c_3, c_4\}\}$. Moreover, by (1) through (4) and Claim 1, it is easy to see that G has a correct embedding shown in Figure 5.4. ¿From this embedding, it is easy to see that the rice-ball layout is correct.

Case 4: $\mathcal{C}^G_{C,F}(\{c_1, c_2, c_3\}) \neq \emptyset$ and $\mathcal{C}^G_{C,F}(\{c_1, c_3, c_4\}) \neq \emptyset$. By Cases 2 and 3, we may assume that $\mathcal{C}^G_{C,F}(\{c_1, c_2, c_4\}) = \mathcal{C}^G_{C,F}(\{c_2, c_3, c_4\}) = \emptyset$. In turn, by Case 1, we may assume that the two nodes $\mathcal{C}^G_{C,F}(\{c_1, c_2, c_3\})$ and $\mathcal{C}^G_{C,F}(\{c_1, c_3, c_4\})$ belong to the same connected component in $\mathcal{G}$. Then, we further distinguish two cases as follows.

Case 4.1: $\mathcal{C}^G_{C,F}(\{c_1, c_3\}) \neq \emptyset$. Then, by (4) and the existence of $\mathcal{L}_p$, the subgraph induced by $C \cup \mathcal{C}^G_{C,F}(\{c_1, c_3\} \cup \mathcal{C}^G_{C,F}(\{c_1, c_2, c_3\} \cup \mathcal{C}^G_{C,F}(\{c_1, c_3, c_4\})$ must have a correct layout shown in Figure 5.5. In Figure 5.5, the point p is touched by exactly one country $d_1 \in \mathcal{C}^G_{C,F}(\{c_1, c_2, c_3\})$ and exactly one country $d_2 \in \mathcal{C}^G_{C,F}(\{c_1, c_3\})$, and $\{c_1, c_3, d_1, d_2\}$ is an MC_4 in G. Similarly, the point q is touched by exactly one country $d_3 \in \mathcal{C}^G_{C,F}(\{c_1, c_3, c_4\})$ and exactly one country $d_4 \in \mathcal{C}^G_{C,F}(\{c_1, c_3\})$, and $\{c_1, c_3, d_3, d_4\}$ is an MC_4 in G. Possibly, $d_2 = d_4$. It is easy to see that we can modify the layout in Figure 5.5 so that c_1 and c_3 also touch p or q without altering its correctness. From the modified layout, it follows that the rice-ball layout is also correct.

Case 4.2: The edge $\{\mathcal{C}^G_{C,F}(\{c_1, c_2, c_3\}), \mathcal{C}^G_{C,F}(\{c_1, c_3, c_4\})\}$ is in $\mathcal{G}$. Then, $\mathcal{C}^G_{C,F}(\{c_1, c_3\}) = \emptyset$. Note that $\mathcal{C}^G_{C,F}(\{c_2, c_4\}) = \emptyset$. Thus, by Claim 1, G has a correct embedding shown in Figure 5.6. In Figure 5.6, exactly one country $d_1 \in \mathcal{C}^G_{C,F}(\{c_1, c_2, c_3\})$ and exactly one country $d_2 \in \mathcal{C}^G_{C,F}(\{c_1, c_3, c_4\})$ touch the point q, and $\{d_1, d_2, c_1, c_3\}$ is an MC_4 in G. It is easy to see that we can modify the layout in Figure 5.6 so that c_1 and c_3 also touch q without altering its correctness. From the modified layout, it follows that the rice-ball layout is also correct.

Case 5: $\mathcal{C}^G_{C,F}(\{c_1, c_2, c_3\}) \neq \emptyset$ and $\mathcal{C}^G_{C,F}(\{c_1, c_2, c_4\}) \neq \emptyset$. Then, in $\mathcal{L}_p$, $\mathcal{C}^G_{C,F}(\{c_1, c_2, c_3\})$ has to touch $\mathcal{C}^G_{C,F}(\{c_1, c_2, c_4\})$. Thus, by (4), $\mathcal{C}^G_{C,F}(\{c_1, c_2\}) = \emptyset$. By Cases 2 and 3, we may assume that $\mathcal{C}^G_{C,F}(\{c_1, c_3, c_4\}) = \emptyset$ and $\mathcal{C}^G_{C,F}(\{c_2, c_3, c_4\}) = \emptyset$. Note that $\mathcal{C}^G_{C,F}(\{c_1, c_3\}) = \emptyset$

and $\mathcal{C}_{C,F}^G(\{c_2, c_4\}) = \emptyset$. Thus, by Claim 1, G has a correct embedding shown in Figure 5.7. From this embedding, it is easy to see that the rice-ball layout is also correct. ∎

By Theorem 3.1, assuming that the rice-ball layout is the best nonpizza layout of C, we can decide whether the rice-ball layout is correct or not in linear time. Suppose that we have found that the rice-ball layout is correct. Then, we can remove C from G as follows.

Case 1: There are W_1 and W_2 in T such that $\{\mathcal{C}_{C,F}^G(W_1), \mathcal{C}_{C,F}^G(W_2)\}$ is an edge in $\mathcal{G}$. Let d_1 be the unique element in $\mathcal{C}_{C,F}^G(W_1)$ with $N_G(d_1) \cap \mathcal{C}_{C,F}^G(W_2) \neq \emptyset$. Similarly, let d_2 be the unique element in $\mathcal{C}_{C,F}^G(W_2)$ with $N_G(d_2) \cap \mathcal{C}_{C,F}^G(W_1) \neq \emptyset$. Let $W_1 \cap W_2 = \{c_i, c_j\}$. Then, it is easy to see that $\langle d_1, c_i, d_2, c_j, d_1\rangle$ is a correct 4-point. By Lemma 1.3, we can remove it from G.

Case 2: There is a $W \in S$ such that the degree of $\mathcal{C}_{C,F}^G(W)$ in $\mathcal{G}$ is 2 and the subgraph of G induced by $\mathcal{C}_{C,F}^G(W)$ is connected. Let $W = \{c_i, c_j\}$, and X be a neighbor of $\mathcal{C}_{C,F}^G(W)$ in $\mathcal{G}$. Let d_1 be the unique element in $\mathcal{C}_{C,F}^G(W)$ with $N_G(d_1) \cap X \neq \emptyset$. Similarly, let d_2 be the unique element in X with $N_G(d_2) \cap \mathcal{C}_{C,F}^G(W) \neq \emptyset$. Then, it is easy to see that $\langle d_1, c_i, d_2, c_j, d_1\rangle$ is a correct 4-point. By Lemma 1.3, we can remove it from G.

Case 3: Neither Case 1 nor Case 2 occur. Then, we construct a new graph $\mathcal{G}'$ from $\mathcal{G}$ as follows. For each $W \in S$ such that the node $\mathcal{C}_{C,F}^G(W)$ has two neighbors X and Y in $\mathcal{G}$ and the subgraph of G induced by $\mathcal{C}_{C,F}^G(W)$ is disconnected, we split the set $\mathcal{C}_{C,F}^G(W)$ into two subsets X' and Y', where X' is the connected component in $G[\mathcal{C}_{C,F}^G(W)]$ with $N_G(X') \cap X \neq \emptyset$ and Y' is the union of the rest connected components in $G[\mathcal{C}_{C,F}^G(W)]$. We delete the node $\mathcal{C}_{C,F}^G(W)$ together with the two edges incident to it from $\mathcal{G}$, and add the two nodes X' and Y' and the two edges $\{X, X'\}$ and $\{Y, Y'\}$ to $\mathcal{G}$. $\mathcal{G}'$ is the resulting graph. Clearly, every connected component in $\mathcal{G}'$ contains at most one node $\mathcal{C}_{C,F}^G(W)$ with $W \in T$. Let K be an arbitrary connected component in $\mathcal{G}'$. Since the rice-ball layout of C is correct, if K contains a node $\mathcal{C}_{C,F}^G(W)$ with $W \in T$, then K must have a correct embedding in which only the three countries in W are on the boundary of the infinite face. Otherwise, K contains only one node and this node is $\mathcal{C}_{C,F}^G(W)$ for some $W \in S$, and K must have a correct embedding in which only the two countries in W are on the boundary of the infinite face. Moreover, given such an embedding of each connected component K in $\mathcal{G}'$, a correct embedding of G can be easily constructed. By Lemma 1.5, such an embedding of each connected component K can be constructed recursively.

5.2 Distinguishing pizzas and 3-type nonpizzas

Assume that the best nonpizza layout of C is a 3-type nonpizza. Our goal is to decide whether there is a correct 3-type nonpizza and to construct one if there is some.

Since the best nonpizza layout of C is an 3-type nonpizza, there are exactly three MC$_4$'s C' in G 3-sharing with C such that the unique element in $C - C'$ and the unique element in $C' - C$ belong to the same connected component in the graph $G' = G - (C \cap C') - \{\{d_1, d_2\} \mid \{d_1, d_2, c_i, c_j\}$ is an MC$_4$ in G for some $\{c_i, c_j\} \subseteq C \cap C'\}$. This follows from Lemma 5.1. Recall that for every $W \subseteq C$ with $|W| = 3$, G has at most one MC$_4$ C' other than C with $W \subseteq C'$. Thus, we may let $C_1 = \{c_1, c_2, c_3, c_5\}$, $C_2 = \{c_1, c_3, c_4, c_6\}$, and $C_3 = \{c_2, c_3, c_4, c_7\}$. Then, the best nonpizza layout of C is unique because c_3' must be c_3 in Figure 5.1(f). Moreover, $\langle c_1, c_3, c_2, c_5, c_1\rangle$, $\langle c_1, c_3, c_4, c_6, c_1\rangle$, and $\langle c_2, c_3, c_4, c_7, c_2\rangle$ each are a correct 4-point. Let $U = \{c_1, c_2, \ldots, c_7\}$, and F be the set of the edges $\{d_1, d_2\}$ in $G - U$ such that $\{d_1, d_2, c_i, c_j\}$ is an MC_4 in G for some $c_i \in U$ and $c_j \in U$. We distinguish two cases as follows.

Case 1: $\{c_5, c_6, c_7\}$ is a clique in G. Then, $C_4 = \{c_3, c_5, c_6, c_7\}$ is an MC_4 in G. If the layout in Figure 5.1(f) is correct, then $G[C_4]$ has a correct nonpizza layout. Thus, by the discussions in § 5.1, we may assume that the best nonpizza layout of $G[C_4]$ is not the rice-ball; otherwise, the layout in Figure 5.1(f) is incorrect. Then, there is a country $c_8 \notin U$ such that $C_5 = \{c_5, c_6, c_7, c_8\}$ is an MC_4 in G and c_3 and c_8 belong to the same connected component in the graph $G' = G - \{c_5, c_6, c_7\} - \{\{d_1, d_2\} \mid \{d_1, d_2, c_i, c_j\}$ is an MC_4 in G for some $\{c_i, c_j\} \subseteq \{c_5, c_6, c_7\}\}$. Moreover, every correct layout of $G[U \cup \{c_8\}]$ in which C is layouted as a nonpizza can be transformed to another of the form in Figure 5.2.1. In Figure 5.2.1, (c'_1, c'_2, c'_4) is a permutation of (c_1, c_2, c_4), and (c'_5, c'_6, c'_7) is a permutation of (c_5, c_6, c_7). Figure 5.2.1 is explicit except that c_8 may touch c'_4. Let K_8 be the connected component in the graph $G - U$ containing c_8. We may assume that $|N_G(K_8) \cap \{c_1, c_2, c_4\}| = 1$; otherwise, the layout in Figure 5.2.1 is incorrect and $G[C]$ has no correct nonpizza layout. We may further assume that $N_G(K_8) \cap \{c_1, c_2, c_4\} = \{c_4\}$; other cases are similar. Then, in Figure 5.2.1, $(c'_1, c'_2, c'_4) = (c_1, c_2, c_4)$ and $(c'_5, c'_6, c'_7) = (c_5, c_6, c_7)$. Let $\mathcal{W}$ be the family of the following sets: $\{c_1, c_3\}$, $\{c_2, c_3\}$, $\{c_4, c_3\}$, $\{c_1, c_5\}$, $\{c_1, c_6\}$, $\{c_2, c_5\}$, $\{c_2, c_7\}$, $\{c_4, c_6\}$, $\{c_4, c_7\}$, $\{c_5, c_6\}$, $\{c_5, c_7\}$, $\{c_1, c_5, c_6\}$, $\{c_2, c_5, c_7\}$, and $\{c_4, c_6, c_7\}$. Then, we claim that the layout of $G[U \cup \{c_8\}]$ in Figure 5.2.1 is correct iff the family $\mathcal{F} = \{\mathcal{C}^G_U(W) \mid W \in \mathcal{W}\}$ is a partition of $V(G) - (U \cup K_8)$. The necessary condition of the claim is obvious. On the other hand, if $\mathcal{F}$ is a partition of $V(G) - (U \cup K_8)$, then every correct pizza layout of $G[U \cup \{c_8\}]$ can be transformed to the one in Figure 5.2.2 which is explicit, and the latter can be transformed to the one in Figure 5.2.1.

Case 2: $\{c_5, c_6, c_7\}$ is not a clique in G. If C has a correct pizza layout, it is easy to see that exactly one of the following (1), (2), and (3) holds:

(1) $\{c_1, c_3, c_5, c_6\}$ and $\{c_2, c_3, c_5, c_7\}$ are MC_4's in G.

(2) $\{c_1, c_3, c_5, c_6\}$ and $\{c_3, c_4, c_6, c_7\}$ are MC_4's in G.

(3) $\{c_3, c_4, c_6, c_7\}$ and $\{c_2, c_3, c_5, c_7\}$ are MC_4's in G.

Thus, if none of (1), (2), and (3) holds, the best nonpizza layout of C must be correct. So, we assume that (1) holds; the other two cases are similar. We then have that $\{c_5, c_6\} \in E(G)$, $\{c_5, c_7\} \in E(G)$, and $\{c_6, c_7\} \notin E(G)$. Moreover, every correct layout of $G[U]$ in which $G[C]$ is layouted as a nonpizza (or pizza, respectively) can be transformed to one of the four layouts in Figure 5.2.3 (respectively, Figure 5.2.4). The eight layouts in Figure 5.2.3 and Figure 5.2.4 are all explicit.

Case 2.1: There is no country $d \notin U$ such that $\{c_1, c_5, c_6, d\}$ or $\{c_2, c_5, c_7, d\}$ is an MC_4 in G. Then, every correct layout of $G[U]$ in which $G[C]$ is layouted as a nonpizza can be transformed to the layout in Figure 5.2.3(a). We may assume that there is no $d \notin U$ such that $\{c_1, c_4, c_6, d\}$ or $\{c_2, c_4, c_7, d\}$ is an MC_4 in G; otherwise, the layout in Figure 5.2.3(a) is incorrect. Then, every correct layout of $G[U]$ in which $G[C]$ is layouted as a pizza can be transformed to the layout in Figure 5.2.4(a). Let F be the set of the edges $\{d_1, d_2\}$ in $G - U$ such that $\{d_1, d_2, c_i, c_j\}$ is an MC_4 in G for some $c_i \in U$ and $c_j \in U$. By considering the maximum extensions of the two layouts in Figure 5.2.3(a) and Figure 5.2.4(a), we can prove that the layout of $G[U]$ in Figure 5.2.3(a) is correct iff $\mathcal{C}^G_{U,F}(\{c_1, c_4\})$, $\mathcal{C}^G_{U,F}(\{c_2, c_4\})$, $\mathcal{C}^G_{U,F}(\{c_3, c_5\})$, $\mathcal{C}^G_{U,F}(\{c_1, c_4, c_6\})$, and $\mathcal{C}^G_{U,F}(\{c_2, c_4, c_7\})$ are all empty.

Case 2.2: There is a country $c_8 \notin U$ such that $\{c_1, c_2, c_4, c_8\}$ is an MC_4 in G. Let

$U_8 = U \cup \{c_8\}$, and F_8 be the set of the edges $\{d_1, d_2\}$ in $G - U_8$ such that $\{d_1, d_2, c_i, c_j\}$ is an MC_4 in G for some $c_i \in U_8$ and $c_j \in U_8$. Then, every correct layout of $G[U_8]$ in which $G[C]$ is layouted as a nonpizza can be transformed to the one in Figure 5.2.5(a). Figure 5.2.5(a) is not explicit but becomes explicit after c_8 is deleted from it. By Figure 5.2.5(a), we may assume that the edges $\{c_5, c_8\}$, $\{c_6, c_8\}$, and $\{c_7, c_8\}$ are all in G; otherwise, C has no correct nonpizza layout. Then, every correct layout of $G[U_8]$ in which $G[C]$ is layouted as a pizza can be transformed to the one in Figure 5.2.5(b). Figure 5.2.5(b) is not explicit but becomes explicit after c_8 is deleted from it. By Figure 5.2.5, if there is a country $d \notin U_8$ such that $\{c_4, c_6, c_8, d\}$ or $\{c_4, c_7, c_8, d\}$ is an MC_4 in G, then the layout in Figure 5.2.5(a) must be correct. Again, by Figure 5.2.5, if there is a country $d \notin U_8$ such that $\{c_5, c_6, c_8, d\}$ or $\{c_5, c_7, c_8, d\}$ is an MC_4 in G, then the layout in Figure 5.2.5(a) is not correct and hence C has no correct nonpizza layout. So, we may assume that there is no country $d \notin U_8$ such that $\{c_4, c_6, c_8, d\}$, $\{c_4, c_7, c_8, d\}$ $\{c_5, c_6, c_8, d\}$, or $\{c_5, c_7, c_8, d\}$ is an MC_4 in G. Consequently, we may assume that Figure 5.2.5(a) and Figure 5.2.5(b) are both explicit. Then, it is easy to prove that the layout in Figure 5.2.5(a) is correct iff $\mathcal{C}^G_{U_8,F_8}(W) = \emptyset$ for each set W among $\{c_1, c_4\}$, $\{c_2, c_4\}$, $\{c_3, c_5\}$, $\{c_5, c_6\}$, $\{c_5, c_7\}$, $\{c_5, c_6, c_8\}$, and $\{c_5, c_7, c_8\}$.

Case 2.3: There is a country $c_8 \notin U$ such that $\{c_1, c_5, c_6, c_8\}$ is an MC_4 in G but there is no country $d \notin U$ such that $\{c_2, c_5, c_7, d\}$ is an MC_4 in G. By Figure 5.2.4, the following (A_1) is a necessary condition for C to have a correct pizza layout:

(A_1) $\{c_4, c_8\} \in E(G)$.

We may assume that (A_1) holds; otherwise, the best nonpizza layout of C must be correct. Then, every correct layout of $G[U]$ in which $G[C]$ is layouted as a nonpizza can be transformed to the one in Figure 5.2.3(b). The following (A_2) is a necessary condition for the layout in Figure 5.2.3(b) to be correct:

(A_2) There is no $d \notin U$ such that $\{c_2, c_4, c_7, d\}$ is an MC_4 in G.

We may assume that (A_2) holds; otherwise, $G[C]$ has no correct nonpizza layout. By this assumption, every correct layout of $G[U]$ in which $G[C]$ is layouted as a pizza can be transformed to the one in Figure 5.2.4(b). Let $U_8 = U \cup \{c_8\}$, and F_8 be the set of the edges $\{d_1, d_2\}$ in $G - U_8$ such that $\{d_1, d_2, c_i, c_j\}$ is an MC_4 in G for some $c_i \in U_8$ and $c_j \in U_8$.

Case 2.3.1: $\{c_7, c_8\} \in E(G)$. We further distinguish two cases as follows.

Case 2.3.1.1: There is some $c_9 \notin U_8$ such that $\{c_5, c_6, c_8, c_9\}$ or $\{c_4, c_6, c_8, c_9\}$ is an MC_4 in G. If $\{c_5, c_6, c_8, c_9\}$ is an MC_4 in G but $\{c_4, c_6, c_8, c_9\}$ is not, then the layout in Figure 5.2.3(b) is incorrect. Similarly, if $\{c_4, c_6, c_8, c_9\}$ is an MC_4 in G but $\{c_5, c_6, c_8, c_9\}$ is not, then the layout in Figure 5.2.4(b) is incorrect. So, we may assume that both $\{c_5, c_6, c_8, c_9\}$ and $\{c_4, c_6, c_8, c_9\}$ are MC_4's in G. Then, every correct layout of $G[U_8 \cup \{c_9\}]$ in which $G[C]$ is layouted as a nonpizza (or pizza, respectively) can be transformed to the layout in Figure 5.2.6(a) (respectively, Figure 5.2.6(b)). The two layouts in Figure 5.2.6 are both explicit. Let $U_9 = U_8 \cup \{c_9\}$, and F_9 be the set of the edges $\{d_1, d_2\}$ in $G - U_9$ such that $\{d_1, d_2, c_i, c_j\}$ is an MC_4 in G for some $c_i \in U_9$ and $c_j \in U_9$. Clearly, the layout in Figure 5.2.6(a) is correct iff $\mathcal{C}^G_{U_9,F_9}(W) = \emptyset$ for every set W among $\{c_1, c_4\}$, $\{c_2, c_4\}$, $\{c_2, c_4, c_7\}$, $\{c_3, c_5\}$, $\{c_4, c_8\}$, $\{c_5, c_6\}$, $\{c_5, c_9\}$, and $\{c_5, c_7, c_9\}$.

Case 2.3.1.2: There is no $d \notin U_8$ such that $\{c_5, c_6, c_8, d\}$ or $\{c_4, c_6, c_8, d\}$ is an MC_4 in G. Then, every correct layout of $G[U_8]$ in which $G[C]$ is layouted as a nonpizza (or pizza, respectively) can be transformed to one of the six layouts in Figure 5.2.7 (respectively, Figure 5.2.8). All the twelve figures in Figure 5.2.7 and Figure 5.2.7 are explicit. The following (B_1) is a necessary condition for each layout in Figure 5.2.7 to be correct:

(B_1) $\mathcal{C}^G_{U_8,F_8}(\{c_5,c_6,c_7,c_8\}) = \emptyset$ and $\mathcal{C}^G_{U_8,F_8}(\{c_2,c_4,c_7,c_8\}) = \emptyset$.

Similarly, the following (B_2) is a necessary condition for each layout in Figure 5.2.8 to be correct:

(B_2) $\mathcal{C}^G_{U_8,F_8}(\{c_4,c_6,c_7,c_8\}) = \emptyset$ and $\mathcal{C}^G_{U_8,F_8}(\{c_2,c_5,c_7,c_8\}) = \emptyset$.

We may assume that B_1 and B_2 both hold; otherwise, we know whether or not the best nonpizza layout of $G[C]$ is correct. Then, every correct layout of $G[U_8]$ in which $G[C]$ is layouted as a nonpizza (or pizza, respectively) can be transformed to one of the first three layouts in Figure 5.2.7 (respectively, Figure 5.2.8). We claim that $G[C]$ has a correct nonpizza layout iff $\mathcal{C}^G_{U_8,F_8}(W) = \emptyset$ for every set W among $\{c_1,c_4\}$, $\{c_2,c_4\}$, $\{c_2,c_4,c_7\}$, $\{c_3,c_5\}$, $\{c_5,c_6\}$, and $\{c_5,c_6,c_8\}$. The necessary condition of the claim follows from the first three layouts in Figure 5.2.7. On the other hand, if $\mathcal{C}^G_{U_8,F_8}(W) = \emptyset$ for each W among the six sets, then the layouts in Figure 5.2.8(a), Figure 5.2.8(b), and Figure 5.2.8(c) can be transformed to the layouts in Figure 5.2.7(a), Figure 5.2.7(b), and Figure 5.2.7(c), respectively.

Case 2.3.2: $\{c_7,c_8\} \notin E(G)$. Then, every correct layout of $G[U_8]$ in which $G[C]$ is layouted as a nonpizza (or pizza, respectively) can be transformed to another of the form in Figure 5.2.9(a) (respectively, Figure 5.2.9(b)). Except that possibly $p_{4,6} = p_{4,8} = p_{6,8}$, Figure 5.2.9(a) is explicit. Except that $p_{5,6} = p_{5,8} = p_{6,8}$, Figure 5.2.9(b) is explicit. If there is no country $d \notin U_8$ such that $\{c_4,c_6,c_8,d\}$ is an MC_4 in G, then similarly to Case 2.1, we can decide whether the best nonpizza layout of C is correct or not. So, suppose that $\{c_4,c_6,c_8,c_9\}$ is an MC_4 in G for some $c_9 \notin U_8$. Let $U_9 = U_8 \cup \{c_9\}$.

Case 2.3.2.1: $\{c_7,c_9\} \in E(G)$. Then similarly to Case 2.3.1, we can decide whether the best nonpizza layout of C is correct or not.

Case 2.3.2.2: $\{c_7,c_9\} \notin E(G)$. Then, every correct layout of $G[U_9]$ in which $G[C]$ is layouted as a nonpizza (or pizza, respectively) can be transformed to another of the form in Figure 5.2.10(a) (respectively, Figure 5.2.10(b)). Except that possibly $p_{5,8} = p_{5,9} = p_{8,9}$, Figure 5.2.10(a) is explicit. Except that $p_{4,8} = p_{4,9} = p_{8,9}$, Figure 5.2.10(b) is explicit. If there is no country $d \notin U_9$ such that $\{c_5,c_8,c_9,d\}$ is an MC_4 in G, then similarly to Case 2.1, we can decide whether the best nonpizza layout of C is correct or not. So, suppose that $\{c_5,c_8,c_9,c_{10}\}$ is an MC_4 in G for some $c_{10} \notin U_9$. Let $U_{10} = U_2 \cup \{c_{10}\}$, and F_{10} be the set of the edges $\{d_1,d_2\}$ in $G - U_{10}$ such that $\{d_1,d_2,c_i,c_j\}$ is an MC_4 in G for some $c_i \in U_{10}$ and $c_j \in U_{10}$.

Case 2.3.2.2.1: $\{c_7,c_{10}\} \in E(G)$. Then similarly to Case 2.3.1, we can decide whether the best nonpizza layout of C is correct or not.

Case 2.3.2.2.2: $\{c_7,c_{10}\} \notin E(G)$. Then, every correct layout of $G[U_{10}]$ in which $G[C]$ is layouted as a nonpizza (or pizza, respectively) can be transformed to another of the form in Figure 5.2.11(a) (respectively, Figure 5.2.11(b)). Except that possibly $p_{4,9} = p_{4,10} = p_{9,10}$, Figure 5.2.11(a) is explicit. Except that $p_{5,9} = p_{5,10} = p_{9,10}$, Figure 5.2.11(b) is explicit. Let $\mathcal{L}_{np}$ be the layout of $G[U_{10}]$ in Figure 5.2.11(a), and $\mathcal{L}_p$ be the layout of $G[U_{10}]$ in Figure 5.2.11(b). By Figure 5.2.11(a), if $\mathcal{L}_{np}$ is correct, then $\mathcal{C}^G_{U_{10}}(W \cup \{c_6\}) = \emptyset$ for all the subsets W of U_{10} except $\{c_1\}$, $\{c_4\}$, and $\{c_8\}$, and $\mathcal{C}^G_{U_{10}}(W \cup \{c_8\}) = \emptyset$ for all the subsets W of U_{10} except $\{c_6\}$, $\{c_5\}$, and $\{c_9\}$. Similarly, by Figure 5.2.11(b), if $\mathcal{L}_p$ is correct, then $\mathcal{C}^G_{U_{10}}(W \cup \{c_6\}) = \emptyset$ for all the subsets W of U_{10} except $\{c_1\}$, $\{c_5\}$, and $\{c_8\}$, and $\mathcal{C}^G_{U_{10}}(W \cup \{c_8\}) = \emptyset$ for all the subsets W of U_{10} except $\{c_6\}$, $\{c_4\}$, and $\{c_9\}$. Thus, we may assume that $\mathcal{C}^G_{U_{10}}(\{c_4, c_6\})$, $\mathcal{C}^G_{U_{10},F_{10}}(\{c_5, c_6\})$, $\mathcal{C}^G_{U_{10}}(\{c_4, c_8\})$, and $\mathcal{C}^G_{U_{10},F_{10}}(\{c_5, c_8\})$ are all empty; otherwise, we know whether $\mathcal{L}_{np}$ is correct or not. Let $X = \{c_6, c_8\} \cup \{\mathcal{C}^G_{U_{10}}(W) \mid W \subseteq U_{10}$ and $W \cap \{c_6, c_8\} \neq \emptyset\}$. Let G' be the graph obtained from G by deleting all the countries in X together with the edges incident to them and further adding the three new edges $\{c_9, c_3\}$, $\{c_9, c_1\}$, and $\{c_{10}, c_1\}$. Let $U' = C \cup \{c_5, c_7, c_9, c_{10}\}$. Figure 5.2.12(a) (or Figure 5.2.12(b), respectively) shows a correct layout of $G'[U']$ when $G[C]$ has a correct nonpizza (respectively, pizza) layout. Except that possibly $p_{4,9} = p_{4,10} = p_{9,10}$, Figure 5.2.12(a) is explicit. Except that $p_{5,9} = p_{5,10} = p_{9,10}$, Figure 5.2.12(b) is explicit. Note that Figure 5.2.12 and Figure 5.2.9 are identical and that distinguishing $\mathcal{L}_{np}$ and $\mathcal{L}_p$ is equivalent to distinguishing the two layouts in Figure 5.2.12. So, we may recursively decide whether the best nonpizza layout of $G'[C]$ is correct or not.

Case 2.4: There is a country $c_8 \notin U$ such that $\{c_2, c_5, c_7, c_8\}$ is an MC_4 in G but there is no country $d \notin U$ such that $\{c_1, c_5, c_6, d\}$ is an MC_4 in G. This case is similar to Case 2.3.

Case 2.5: There are two distinct countries $c_8 \notin U$ and $c_9 \notin U$ such that $\{c_1, c_5, c_6, c_8\}$ and $\{c_2, c_5, c_7, c_9\}$ are MC_4's in G. By Figure 5.2.4, the following (A_3) is a necessary condition for C to have a correct pizza layout:

(A_3) $\{c_8, c_4\} \in E(G)$ and $\{c_9, c_4\} \in E(G)$.

We may assume that (A_3) holds; otherwise, the best nonpizza layout of C must be correct. Then, every correct layout of $G[U]$ in which $G[C]$ is layouted as a nonpizza (or pizza, respectively) can be transformed to the one in Figure 5.2.3(d) (respectively, Figure 5.2.4(d)). Let $U_9 = U \cup \{c_8, c_9\}$, and F_9 be the set of the edges $\{d_1, d_2\}$ in $G - U_9$ such that $\{d_1, d_2, c_i, c_j\}$ is an MC_4 in G for some $c_i \in U_9$ and $c_j \in U_9$.

Case 2.5.1: $\{c_8, c_9\} \in E(G)$. We further distinguish four cases as follows.

Case 2.5.1.1: There is some $c_{10} \notin U_9$ such that $\{c_5, c_6, c_8, c_{10}\}$ or $\{c_4, c_6, c_8, c_{10}\}$ is an MC_4 in G, and there is some $c_{11} \notin U_9$ such that $\{c_5, c_7, c_9, c_{11}\}$ or $\{c_4, c_7, c_9, c_{11}\}$ is an MC_4 in G. If (1) $\{c_5, c_6, c_8, c_{10}\}$ is an MC_4 in G but $\{c_4, c_6, c_8, c_{10}\}$ is not or (2) $\{c_5, c_7, c_9, c_{11}\}$ is an MC_4 in G but $\{c_4, c_7, c_9, c_{11}\}$ is not, then the layout in Figure 5.2.3(d) is incorrect. Similarly, If (1) $\{c_4, c_6, c_8, c_{10}\}$ is an MC_4 in G but $\{c_5, c_6, c_8, c_{10}\}$ is not or (2) $\{c_4, c_7, c_9, c_{11}\}$ is an MC_4 in G but $\{c_5, c_7, c_9, c_{11}\}$ is not, then the layout in Figure 5.2.4(d) is incorrect. So, we may assume that $\{c_4, c_6, c_8, c_{10}\}$, $\{c_5, c_6, c_8, c_{10}\}$, $\{c_4, c_7, c_9, c_{11}\}$, and $\{c_5, c_7, c_9, c_{11}\}$ are MC_4's in G. Then, c_{10} and c_{11} must be the same country and every correct layout of $G[U_9 \cup \{c_{10}\}]$ in which $G[C]$ is layouted as a nonpizza (or pizza, respectively) can be transformed to the layout in Figure 5.2.13(a) (respectively, Figure 5.2.13(b)). The two layouts in Figure 5.2.13 are both explicit. Let $U_{10} = U_9 \cup \{c_{10}\}$. Clearly, the layout in Figure 5.2.13(a) is correct iff $\mathcal{C}^G_{U_{10}}(W) = \emptyset$ for every set W among $\{c_1, c_4\}$, $\{c_2, c_4\}$, $\{c_3, c_5\}$, $\{c_4, c_8\}$, $\{c_4, c_9\}$, $\{c_5, c_6\}$, $\{c_5, c_7\}$, and $\{c_5, c_{10}\}$.

Case 2.5.1.2: There is some $c_{10} \notin U_9$ such that $\{c_5, c_6, c_8, c_{10}\}$ or $\{c_4, c_6, c_8, c_{10}\}$ is an MC_4 in G, but there is no $d \notin U_9$ such that $\{c_5, c_7, c_9, d\}$ or $\{c_4, c_7, c_9, d\}$ is an MC_4 in G. If $\{c_5, c_6, c_8, c_{10}\}$ is an MC_4 in G but $\{c_4, c_6, c_8, c_{10}\}$ is not, then the layout in Figure 5.2.3(d) is incorrect. Similarly, if $\{c_4, c_6, c_8, c_{10}\}$ is an MC_4 in G but $\{c_5, c_6, c_8, c_{10}\}$ is not, then the layout in Figure 5.2.4(d) is incorrect. So, we may assume that both $\{c_4, c_6, c_8, c_{10}\}$ and $\{c_5, c_6, c_8, c_{10}\}$ are MC_4's in G. Then, every correct layout of $G[U_9 \cup \{c_{10}\}]$ in which $G[C]$ is layouted as a nonpizza (or pizza, respectively) can be transformed to the layout in Figure 5.2.14(a) (respectively, Figure 5.2.14(b)). Both the two figures in Figure 5.2.14 are explicit. Let $U_{10} = U_9 \cup \{c_{10}\}$, and F_{10} be the set of the edges $\{d_1, d_2\}$ in $G - U_{10}$ such that $\{d_1, d_2, c_i, c_j\}$ is an MC_4 in G for some $c_i \in U_{10}$ and $c_j \in U_{10}$. Clearly, the layout in Figure 5.2.14(a) is correct iff $\mathcal{C}^G_{U_{10},F_{10}}(W) = \emptyset$ for every set W among $\{c_1, c_4\}$, $\{c_2, c_4\}$, $\{c_3, c_5\}$, $\{c_4, c_8\}$, $\{c_5, c_6\}$, $\{c_5, c_7\}$, $\{c_5, c_{10}\}$, $\{c_5, c_7, c_9\}$, and $\{c_5, c_9, c_{10}\}$.

Case 2.5.1.3: There is some $c_{10} \notin U_9$ such that $\{c_4, c_7, c_9, c_{10}\}$ or $\{c_5, c_7, c_9, c_{10}\}$ is an MC_4 in G, but there is no $d \notin U_9$ such that $\{c_4, c_6, c_8, d\}$ or $\{c_5, c_6, c_8, d\}$ is an MC_4 in G. This case is similar to Case 2.5.1.2.

Case 2.5.1.4: There is no $d \notin U_9$ such that $\{c_4, c_6, c_8, d\}$, $\{c_5, c_6, c_8, d\}$, $\{c_4, c_7, c_9, d\}$, or $\{c_5, c_7, c_9, d\}$ is an MC_4 in G. Then, every correct layout of $G[U_9]$ in which $G[C]$ is layouted as a nonpizza (or pizza, respectively) can be transformed to one of the five layouts in Figure 5.2.15 (respectively, Figure 5.2.16). All the ten figures in Figure 5.2.15 and Figure 5.2.16 are explicit. The following (B_3) is a necessary condition for each layout in Figure 5.2.15 to be correct:

(B_3) $\mathcal{C}^G_{U_9,F_9}(\{c_5, c_6, c_8, c_9\}) = \emptyset$ and $\mathcal{C}^G_{U_9,F_9}(\{c_5, c_7, c_8, c_9\}) = \emptyset$.

Similarly, the following (B_4) is a necessary condition for each layout in Figure 5.2.16 to be correct:

(B_4) $\mathcal{C}^G_{U_9,F_9}(\{c_4, c_6, c_8, c_9\}) = \emptyset$ and $\mathcal{C}^G_{U_9,F_9}(\{c_4, c_7, c_8, c_9\}) = \emptyset$.

We may assume that B_3 and B_4 both hold; otherwise, we know whether or not the best nonpizza layout of $G[C]$ is correct. Then, every correct layout of $G[U_9]$ in which $G[C]$ is layouted as a nonpizza (or pizza, respectively) can be transformed to one of the first three layouts in Figure 5.2.15 (respectively, Figure 5.2.16). We claim that $G[C]$ has a correct nonpizza layout iff $\mathcal{C}^G_{U_9,F_9}(W) = \emptyset$ for every set W among $\{c_1, c_4\}$, $\{c_2, c_4\}$, $\{c_3, c_5\}$, $\{c_5, c_6\}$, $\{c_5, c_7\}$, $\{c_5, c_6, c_8\}$, and $\{c_5, c_7, c_9\}$. The necessary condition of the claim follows from the first three layouts in Figure 5.2.15. On the other hand, if $\mathcal{C}^G_{U_9,F_9}(W) = \emptyset$ for each W among the seven sets, then the layouts in Figure 5.2.16(a), Figure 5.2.16(b), and Figure 5.2.16(c) can be transformed to the layouts in Figure 5.2.15(a), Figure 5.2.15(b), and Figure 5.2.15(c), respectively.

Case 2.5.2: $\{c_8, c_9\} \notin E(G)$. Then, every correct layout of $G[U_9]$ in which $G[C]$ is layouted as a nonpizza (or pizza, respectively) can be transformed to another of the form in Figure 5.2.17(a) (respectively, Figure 5.2.17(b)). Except that possibly $p_{4,6} = p_{4,8} = p_{6,8}$ or $p_{4,7} = p_{4,9} = p_{7,9}$, Figure 5.2.17(a) is explicit. Except that possibly $p_{5,6} = p_{5,8} = p_{6,8}$ or $p_{5,7} = p_{5,9} = p_{7,9}$, Figure 5.2.17(b) is explicit. If there is no country $d \notin U_9$ such that $\{c_4, c_6, c_8, d\}$ or $\{c_4, c_7, c_9, d\}$ is an MC_4 in G, then similarly to Case 2.1, we can decide whether the best nonpizza layout of $G[C]$ is correct or not. Thus, it suffices to consider the following four cases:

Case 2.5.2.1: There is no country $d \notin U_9$ such that $\{c_4, c_7, c_9, d\}$ is an MC_4 in G. We may assume that there is no $d \notin U_9$ such that $\{c_5, c_7, c_9, d\}$ is an MC_4 in G; otherwise, $G[C]$ has no correct nonpizza layout. In turn, we can assume that the three points $p_{4,7}$, $p_{4,9}$, and $p_{7,9}$ in Figure 5.2.17(a) are distinct and that the three points $p_{5,7}$, $p_{5,9}$, and $p_{7,9}$ in Figure 5.2.17(b) are distinct. Let $\mathcal{L}_{np}$ be the layout of $G[U_9]$ in Figure 5.2.17(a), and $\mathcal{L}_p$ be the layout of $G[U_9]$ in Figure 5.2.17(b). Let $\mathcal{E}_{np}$ (or $\mathcal{E}_p$, respectively) be the maximum extension of $\mathcal{L}_{np}$ (respectively, $\mathcal{L}_p$) if $\mathcal{L}_{np}$ (respectively, $\mathcal{L}_p$) is correct. If $\mathcal{L}_{np}$ is correct, then from $\mathcal{E}_{np}$, it is easy to see that $\mathcal{C}^G_{U_9,F_9}(W \cup \{c_7\}) = \emptyset$ for all the subsets W of U_9 except $\{c_2\}$, $\{c_4\}$, $\{c_9\}$, and $\{c_4, c_9\}$. Similarly, if $\mathcal{L}_p$ is correct, then from $\mathcal{E}_p$, it is easy to see that $\mathcal{C}^G_{U_9,F_9}(W \cup \{c_7\}) = \emptyset$ for all the subsets W of U_9 except $\{c_2\}$, $\{c_5\}$, $\{c_9\}$, and $\{c_5, c_9\}$. Thus, we may assume that $\mathcal{C}^G_{U_9,F_9}(\{c_4, c_7\})$, $\mathcal{C}^G_{U_9,F_9}(\{c_5, c_7\})$, $\mathcal{C}^G_{U_9,F_9}(\{c_4, c_7, c_9\})$, and $\mathcal{C}^G_{U_9,F_9}(\{c_5,c_7,c_9\})$ are all empty; otherwise, we know whether $\mathcal{L}_{np}$ is correct or not. Let $X = \{c_7\} \cup \{\mathcal{C}^G_{U_9,F_9}(W) \mid W \subseteq U_9$ and $c_7 \in W\}$. Let G' be the graph obtained from G by deleting all the countries in X together with the edges incident to them and adding the the new edge $\{c_9, c_3\}$. Let $U' = C \cup \{c_5, c_6, c_8, c_9\}$. Figure 5.2.18(a) (or Figure 5.2.18(b), respectively) shows a correct layout of $G'[U']$ when $G[C]$ has a correct nonpizza (respectively, pizza) layout. Except that possibly $p_{4,6} = p_{4,8} = p_{6,8}$, Figure 5.2.18(a) is explicit. Except that $p_{5,6} = p_{5,8} = p_{6,8}$, Figure 5.2.18(b) is explicit. Note that Figure 5.2.18 and Figure 5.2.9 are identical and that distinguishing $\mathcal{L}_{np}$ and $\mathcal{L}_p$ is equivalent to distinguishing the two layouts in Figure 5.2.18. So, we may recursively decide whether the best nonpizza layout of $G'[C]$ is correct or not.

Case 2.5.2.2: There is no country $d \notin U_9$ such that $\{c_4, c_6, c_8, d\}$ is an MC$_4$ in G. This case is similar to Case 2.5.2.1.

Case 2.5.2.3: There is a country $c_{10} \notin U_9$ such that both $\{c_4, c_7, c_9, c_{10}\}$ and $\{c_4, c_6, c_8, d\}$ are MC$_4$'s in G. Then, similarly to Case 2.2, we can decide whether the best nonpizza layout of $G[C]$ is correct or not.

Case 2.5.2.4: There are two distinct countries $c_{10} \notin U_9$ and $c_{11} \notin U_9$ such that both $\{c_4, c_6, c_8, c_{10}\}$ and $\{c_4, c_7, c_9, c_{11}\}$ are MC$_4$'s in G. Let $U_{11} = U_9 \cup \{c_{10}, c_{11}\}$. We may assume that both $\{c_5, c_{10}\}$ and $\{c_5, c_{11}\}$ are edges in G; otherwise, C has no correct pizza layout.

Case 2.5.2.4.1: $\{c_{10}, c_{11}\} \in E(G)$. Then, similarly to Case 2.5.1, we can decide whether the best nonpizza layout of C is correct or not.

Case 2.5.2.4.2: $\{c_{10}, c_{11}\} \notin E(G)$. Then, every correct layout of $G[U_{11}]$ in which $G[C]$ is layouted as a nonpizza (or pizza, respectively) can be transformed to another of the form in Figure 5.2.19(a) (respectively, Figure 5.2.19(b)). Except that possibly $p_{5,8} = p_{5,10} = p_{8,10}$ or $p_{5,9} = p_{5,11} = p_{9,11}$, Figure 5.2.19(a) is explicit. Except that possibly $p_{4,8} = p_{4,10} = p_{8,10}$ or $p_{4,9} = p_{4,11} = p_{9,11}$, Figure 5.2.19(b) is explicit. We may assume that there are two distinct countries $c_{12} \notin U_{11}$ and $c_{13} \notin U_{11}$ such that both $\{c_5, c_8, c_{10}, c_{12}\}$ and $\{c_5, c_9, c_{11}, c_{13}\}$ are MC$_4$'s in G; otherwise, similarly to Case 2.1, Case 2.5.2.1, or Case 2.2, we can decide whether the best nonpizza layout of C is correct or not. Let $U_{13} = U_{11} \cup \{c_{12}, c_{13}\}$.

Case 2.5.2.4.2.1: $\{c_{12}, c_{13}\} \in E(G)$. Then, similarly to Case 2.5.1, we can decide whether the best nonpizza layout of C is correct or not.

Case 2.5.2.4.2.2: $\{c_{12}, c_{13}\} \notin E(G)$. Then, every correct layout of $G[U_{13}]$ in which $G[C]$ is layouted as a nonpizza (or pizza, respectively) can be transformed to another of the form in Figure 5.2.20(a) (respectively, Figure 5.2.20(b)). Except that possibly $p_{4,10} = p_{4,12} = p_{10,12}$ or $p_{4,11} = p_{4,13} = p_{11,13}$, Figure 5.2.20(a) is explicit. Except that possibly $p_{5,10} = p_{5,12} = p_{10,12}$ or $p_{5,11} = p_{5,13} = p_{11,13}$, Figure 5.2.20(b) is explicit. Let $\mathcal{L}_{np}$ be the layout of $G[U_{13}]$ in Figure 5.2.20(a), and $\mathcal{L}_p$ be the layout of $G[U_{13}]$ in Figure 5.2.20(b). By Figure 5.2.20(a), if $\mathcal{L}_{np}$ is correct, then $\mathcal{C}^G_{U_{13}}(W \cup \{c_6\}) = \emptyset$ for all the subsets W of U_{13} except $\{c_1\}$, $\{c_4\}$, and $\{c_8\}$, $\mathcal{C}^G_{U_{13}}(W \cup \{c_8\}) = \emptyset$ for all the subsets W of U_{13} except $\{c_6\}$, $\{c_5\}$, and $\{c_{10}\}$, $\mathcal{C}^G_{U_{13}}(W \cup \{c_7\}) = \emptyset$ for all the subsets W of U_{13} except $\{c_2\}$, $\{c_4\}$, and $\{c_9\}$, and $\mathcal{C}^G_{U_{13}}(W \cup \{c_9\}) = \emptyset$ for all the subsets W of U_{13} except $\{c_7\}$, $\{c_5\}$, and $\{c_{11}\}$. Similarly, by Figure 5.2.20(b), if $\mathcal{L}_p$ is correct, then $\mathcal{C}^G_{U_{13}}(W \cup \{c_6\}) = \emptyset$ for all the subsets W of U_{13} except $\{c_1\}$, $\{c_5\}$, and $\{c_8\}$, $\mathcal{C}^G_{U_{13}}(W \cup \{c_8\}) = \emptyset$ for all the subsets W of U_{13} except $\{c_6\}$, $\{c_4\}$, and $\{c_{10}\}$, $\mathcal{C}^G_{U_{13}}(W \cup \{c_7\}) = \emptyset$ for all the subsets W of U_{13} except $\{c_2\}$, $\{c_5\}$, and $\{c_9\}$, and $\mathcal{C}^G_{U_{13}}(W \cup \{c_9\}) = \emptyset$ for all the subsets W of U_{13} except $\{c_7\}$, $\{c_4\}$, and $\{c_{11}\}$.

Thus, we may assume that $\mathcal{C}^G_{U_{13}}(\{c_4, c_6\})$, $\mathcal{C}^G_{U_{13},F}(\{c_5, c_6\})$, $\mathcal{C}^G_{U_{13}}(\{c_4, c_8\})$, $\mathcal{C}^G_{U_{13},F}(\{c_5, c_8\})$, $\mathcal{C}^G_{U_{13}}(\{c_4, c_7\})$, $\mathcal{C}^G_{U_{13},F}(\{c_5, c_7\})$, and $\mathcal{C}^G_{U_{13}}(\{c_4, c_9\})$, $\mathcal{C}^G_{U_{13},F}(\{c_5, c_9\})$ are all empty; otherwise, we know whether $\mathcal{L}_{np}$ is correct or not. Let $X = \{c_6, \ldots, c_9\} \cup \{\mathcal{C}^G_{U_{13}}(W) \mid W \subseteq U_{13}$ and $W \cap \{c_6, \ldots, c_9\} \neq \emptyset\}$. Let G' be the graph obtained from G by deleting all the countries in X together with the edges incident to them and further adding the six new edges $\{c_{10}, c_1\}$, $\{c_{10}, c_3\}$, $\{c_{12}, c_1\}$, $\{c_{11}, c_2\}$, $\{c_{11}, c_3\}$, and $\{c_{13}, c_2\}$. Let $U' = C \cup \{c_5, c_{10}, \ldots, c_{13}\}$. Figure 5.2.21(a) (or Figure 5.2.21(b), respectively) shows a correct layout of $G'[U']$ when $G[C]$ has a correct nonpizza (respectively, pizza) layout. Except that possibly $p_{4,10} = p_{4,12} = p_{10,12}$ or $p_{4,11} = p_{4,13} = p_{11,13}$, Figure 5.2.21(a) is explicit. Except that possibly $p_{5,10} = p_{5,12} = p_{10,12}$ or $p_{5,11} = p_{5,13} = p_{11,13}$, Figure 5.2.21(b) is explicit. Note that Figure 5.2.21 and Figure 5.2.17 are identical and that distinguishing $\mathcal{L}_{np}$ and $\mathcal{L}_p$ is equivalent to distinguishing the two layouts in Figure 5.2.21. So, we may recursively decide whether the best nonpizza layout of $G'[C]$ is correct or not.

5.3 Distinguishing pizzas and 2-type nonpizzas

Assume that the best nonpizza layout of C is a 2-type nonpizza. Our goal is to decide whether there is a correct 2-type nonpizza and to construct one if there is some. By the discussions in § 5.1 and § 5.2 we may assume that every MC_4 in G whose best nonpizza layout is the rice-ball or of 3-type has no correct nonpizza layout. Since the best nonpizza layout of C is a 2-type nonpizza, there are exactly two MC_4's C' in G 3-sharing with C such that the unique element in $C - C'$ and the unique element in $C' - C$ belong to the same connected component in the graph $G' = G - (C \cap C') - \{\{d_1, d_2\} \mid \{d_1, d_2, c_i, c_j\}$ is an MC_4 in G for some $\{c_i, c_j\} \subseteq C \cap C'\}$. This follows from Lemma 5.1. Let the two MC_4's be $C_1 = \{c_1, c_2, c_3, c_5\}$ and $C_2 = \{c_2, c_3, c_4, c_6\}$. Let $U = \{c_1, c_2, \ldots, c_6\}$, and F be the set of the edges $\{d_1, d_2\}$ in $G - U$ such that $\{d_1, d_2, c_i, c_j\}$ is an MC_4 in G for some $c_i \in U$ and $c_j \in U$.

5.3.1 The case where $\{c_5, c_6\} \in E(G)$

Assume that $\{c_5, c_6\} \in E(G)$. Then, every correct layout of $G[U]$ in which C is layouted as a nonpizza can be transformed to the one in Figure 5.3.1(a) or the one in Figure 5.3.1(b). Both figures in Figure 5.3.1 are explicit, and (c'_2, c'_3) is a permutation of (c_2, c_3) in both of them.

Lemma 5.4 Suppose that there is no $d \notin U$ such that $\{d, c_2, c_5, c_6\}$ or $\{d, c_3, c_5, c_6\}$ is an MC_4 in G. Then, for every permutation (c'_2, c'_3) of (c_2, c_3), the layout of $G[U]$ in Figure 5.3.1(a) is correct iff $\mathcal{C}^G_{U,F}(W) = \emptyset$ for every $W \subseteq U$ such that $\{c_1, c'_2\} \subseteq W$, $\{c'_2, c_4\} \subseteq W$, $\{c'_3, c_5\} \subseteq W$, or $\{c'_3, c_6\} \subseteq W$.
Proof. Since there is no $d \notin U$ such that $\{d, c_2, c_5, c_6\}$ or $\{d, c_3, c_5, c_6\}$ is an MC_4 in G, the layout in Figure 5.3.1(b) can be transformed to the one in Figure 5.3.1(a). Fix a permutation (c'_2, c'_3) of (c_2, c_3). The necessary condition is clear from Figure 5.3.1(a). To show the sufficient condition, we assume that $\mathcal{C}^G_{U,F}(W) = \emptyset$ for every $W \subseteq U$ satisfying the condition in the lemma. Then, every correct layout of $G[U]$ in which $G[C]$ is layouted as a pizza can be transformed to one of the three layouts in Figure 5.3.2. It is clear that each of the three layouts can be transformed to the layout in Figure 5.3.1(a). Moreover, the layout

obtained from the one in Figure 5.3.1(a) by exchanging the positions of c_2' and c_3' can also be transformed to the layout in Figure 5.3.1(a). ∎

By Figure 5.3.1, we may assume that there are no $d_1 \notin U$ and $d_2 \notin U$ (possibly $d_1 = d_2$) such that both $\{c_2, c_5, c_6, d_1\}$ and $\{c_3, c_5, c_6, d_2\}$ are MC_4's in G; otherwise, C has no correct nonpizza layout. Thus, by Lemma 5.4 and symmetry, we may assume that there is some $c_7 \notin U$ such that $\{c_2, c_5, c_6, c_7\}$ is an MC_4 in G. Then, we can hereafter assume that the permutation (c_2', c_3') in Figure 5.3.1 is (c_2, c_3). Let $U_7 = U \cup \{c_7\}$, and F_7 be the set of the edges $\{d_1, d_2\}$ in $G - U_7$ such that $\{d_1, d_2, c_i, c_j\}$ is an MC_4 in G for some $c_i \in U_7$ and $c_j \in U_7$. We distinguish four cases as follows.

Case 1: $\{c_1, c_7\} \in E(G)$ but $\{c_4, c_7\} \notin E(G)$. Then, every correct layout of $G[U]$ in which $G[C]$ is layouted as a nonpizza can be transformed to the one in Figure 5.3.1(b). Note that $C_3 = \{c_1, c_2, c_5, c_7\}$ is an MC_4 in G. If the layout in Figure 5.3.1(b) is correct, then $G[C_3]$ has a correct nonpizza layout. Thus, we may assume that the best nonpizza layout of $G[C_3]$ is not of 3-type; otherwise, the layout in Figure 5.3.1(b) is incorrect. Then, every correct nonpizza layout of $G[U_7]$ can be transformed to the one in Figure 5.3.3(a). Figure 5.3.3(a) is explicit. We claim that the layout in Figure 5.3.3(a) is correct iff $\mathcal{C}^G_{U_7,F_7}(W) = \emptyset$ for every $W \subseteq U_7$ such that $\{c_1, c_2\} \subseteq W$, $\{c_2, c_4\} \subseteq W$, $\{c_2, c_7\} \subseteq W$, $\{c_3, c_5\} \subseteq W$, $\{c_3, c_6\} \subseteq W$, or $\{c_5, c_6\} \subseteq W$. This can be seen by noting that whenever $\mathcal{C}^G_{U_7,F_7}(W) = \emptyset$ for every such subset W of U_7, every correct layout of $G[U_7]$ in which C is layouted as a pizza can be transformed to one of the last two layouts in Figure 5.3.3, both of which can be transformed to the layout in Figure 5.3.3(a).

Case 2: $\{c_4, c_7\} \in E(G)$ but $\{c_1, c_7\} \notin E(G)$. This case is similar to Case 1.

Case 3: $\{c_4, c_7\} \in E(G)$ and $\{c_1, c_7\} \in E(G)$. Then, every correct layout of $G[U]$ in which $G[C]$ is layouted as a nonpizza can be transformed to the one in Figure 5.3.1(b). Note that $C_4 = \{c_1, c_2, c_4, c_7\}$ is an MC_4 in G. If the layout in Figure 5.3.1(b) is correct, then $G[C_4]$ has a correct nonpizza layout. Thus, we may assume that the best nonpizza layout of $G[C_4]$ is not the rice-ball; otherwise, the layout in Figure 5.3.1(b) is incorrect. Then, there is some $c_8 \notin U_7$ such that $\{c_1, c_4, c_7, c_8\}$ is an MC_4 in G, and every correct layout of $[U_8 \cup \{c_8\}]$ in which C is layouted as a nonpizza can be transformed to one of the three layouts in Figure 5.3.4. All the three figures in Figure 5.3.4 are explicit except that the country c_8 may touch c_5 in Figure 5.3.4(a), may touch c_6 in Figure 5.3.4(b), and may touch c_3 in Figure 5.3.4(c). Let K_8 be the connected component in the graph $G - U_7 - F_7$ containing c_8. Since the best nonpizza layout of $G[C_4]$ is not the rice-ball, $c_5 \in N_G(K_8)$ if the layout in Figure 5.3.4(a) is correct, $c_6 \in N_G(K_8)$ if the layout in Figure 5.3.4(b) is correct, and $c_3 \in N_G(K_8)$ if the layout in Figure 5.3.4(c) is correct. Thus, if $N_G(K_8) \cap \{c_3, c_5, c_6\} = \emptyset$, then $G[C]$ has no correct nonpizza layout. So, we assume that $N_G(K_8) \cap \{c_3, c_5, c_6\} \neq \emptyset$. Let $U_8 = U_7 \cup \{c_8\}$.

Case 3.1: $c_5 \in N_G(K_8)$. Then, by Figure 5.3.4, the last two layouts in Figure 5.3.4 cannot be correct. We claim that the layout of $G[U_8]$ in Figure 5.3.4(a) is correct iff $\mathcal{C}^G_{U_7,F_7}(W) = \emptyset$ for every $W \subseteq U$ such that $\{c_2, c_4\} \subseteq W$, $\{c_3, c_5\} \subseteq W$, or $\{c_5, c_6\} \subseteq W$. This can be seen by noting that whenver $\mathcal{C}^G_{U_7,F_7}(W) = \emptyset$ for every such subset W of U, every correct layout of $G[U_8]$ can be transformed to the layout in Figure 5.3.5, which can then be transformed to the layout in Figure 5.3.4(a).

Case 3.2: $c_6 \in N_G(K_8)$. Similar to Case 3.1.

Case 3.3: $c_3 \in N_G(K_8)$. Then, we can prove that $G[C]$ has no correct pizza layout. This can be seen by trying every possible pizza layout of $G[C]$.

Case 4: $\{c_4, c_7\} \notin E(G)$ and $\{c_1, c_7\} \notin E(G)$. Then, every correct layout of $G[U_7]$ can be transformed to one of the two layouts in Figure 5.3.6. Figure 5.3.6(b) is explicit and Figure 5.3.6(a) becomes explicit after c_7 is removed from it.

Case 4.1: $c_7 \in \mathcal{C}^G_{U,F}(\{c_2, c_5, c_6\})$. Then, the layout in Figure 5.3.6(b) can be transformed to the layout in Figure 5.3.6(a). Similarly to Lemma 5.4, we can decide whether the layout of $G[U]$ obtained from Figure 5.3.6(a) by deleting c_7 is really correct or not.

Case 4.2: $c_7 \notin \mathcal{C}^G_{U,F}(\{c_2, c_5, c_6\})$. Then, the layout in Figure 5.3.6(a) cannot be correct. We claim that the layout in Figure 5.3.6(b) is correct iff $\mathcal{C}^G_{U_7,F_7}(W) = \emptyset$ for every $W \subseteq U$ such that $\{c_3, c_5\} \subseteq W$ or $\{c_5, c_6\} \subseteq W$. This can be seen by noting that $G[C]$ cannot have a correct pizza layout if $\{c_4, c_7\} \notin E(G)$, $\{c_1, c_7\} \notin E(G)$, and $c_7 \notin \mathcal{C}^G_{U,F}(\{c_2, c_5, c_6\})$.

5.3.2 The case where $\{c_5, c_6\} \notin E(G)$

Assume that $\{c_5, c_6\} \notin E(G)$. Then, every correct layout of $G[U]$ in which C is layouted as a nonpizza can be transformed to another of one of the two forms in Figure 5.3.7. In both forms, (c'_2, c'_3) is a permutation of (c_2, c_3). Both figures in Figure 5.3.7 are explicit. To distinguish the two forms, we need the following two facts.

Fact 5 Let $U_5 = C \cup \{c_5\}$, and F_5 be the set of the edges $\{d_1, d_2\}$ in $G - U_5$ such that $\{d_1, d_2, c_i, c_j\}$ is an MC$_4$ in G for some $\{c_i, c_j\} \subseteq U_5$. Then, $\mathcal{C}^G_{U_5,F_5}(W \cup \{c_4, c_5\}) \neq \emptyset$ for some $W \subseteq U_5$.

Proof. Let K be the connected component of the graph $G' = G - \{c_1, c_2, c_3\} - \{\{d_1, d_2\} \mid \{d_1, d_2, c_i, c_j\}$ is an MC$_4$ in G for some $\{c_i, c_j\} \subseteq \{c_1, c_2, c_3\}\}$ containing both c_4 and c_5. For every edge $\{d_1, d_2\}$ in $K - \{c_4, c_5\}$ such that $\{d_1, d_2, c_i, c_4\}$ is an MC$_4$ in G for some $c_i \in \{c_1, c_2, c_3\}$, c_4 and c_5 still belong to the same connected component in the graph $K - \{\{d_1, d_2\}\}$. Similarly, for every edge $\{d_1, d_2\}$ in $K - \{c_4, c_5\}$ such that $\{d_1, d_2, c_i, c_5\}$ is an MC$_4$ in G for some $c_i \in \{c_1, c_2, c_3\}$, c_4 and c_5 still belong to the same connected component in the graph $K - \{\{d_1, d_2\}\}$. Consequently, c_4 and c_5 still belong to the same connected component (say, K') in the graph $K - F$. Obviously, there is a connected component K'' in the graph $K' - \{c_4, c_5\}$ such that $\{c_4, c_5\} \subseteq N_G(K'')$. Since K'' is also a connected component in $G - U_5 - F_5$, the fact follows. ∎

Fact 6 Let $U_6 = C \cup \{c_6\}$, and F_6 be the set of the edges $\{d_1, d_2\}$ in $G - U_6$ such that $\{d_1, d_2, c_i, c_j\}$ is an MC$_4$ in G for some $\{c_i, c_j\} \subseteq U_6$. Then, $\mathcal{C}^G_{U_6,F_6}(W \cup \{c_1, c_6\}) \neq \emptyset$ for some $W \subseteq U_6$.

Let F_4 be the set of the edges $\{d_1, d_2\}$ in $G - C$ such that $\{d_1, d_2, c_1, c_4\}$ is an MC$_4$ in G. Let K_5 (or K_6, respectively) be the connected component in the graph $G - C - F_4$ containing c_5 (respectively, c_6). By the above two facts, $G[U]$ has no correct layout of the form in Figure 5.3.7(a) if $K_5 = K_6$, and has no correct layout of the form in Figure 5.3.7(b) if $N_G(K_5) \cap K_6 = \emptyset$.

Lemma 5.5 Assume that $G[C]$ has a correct nonpizza layout. Then, for every permutation (c'_1, c'_2) of (c_1, c_2), the following (1), (2), and (3) hold:

(1) When $N_G(K_5) \cap K_6 = \emptyset$, the layout of $G[U]$ in Figure 5.3.7(a) is correct iff

(1.1) $c'_3 \notin N_G(K_5 - \{c_5\})$ and $c'_2 \notin N_G(K_6 - \{c_6\})$,

(1.2) $\mathcal{C}^G_{C,F_4}(\{c_1, c'_2\}) = \mathcal{C}^G_{C,F_4}(\{c_4, c'_3\}) = \emptyset$, and

(1.3) $N_G(K_5) \cap \mathcal{C}^G_{C,F_4}(\{c_1, c_4, c'_3\}) = N_G(K_6) \cap \mathcal{C}^G_{C,F_4}(\{c_1, c_4, c'_2\}) = \emptyset$.

(2) When $K_5 = K_6$, the layout of $G[U]$ in Figure 5.3.7(b) is correct iff $c'_3 \notin N_G(K_5 - \{c_5, c_6\})$ and the family $\{\mathcal{C}^G_{C,F_4}(W) \mid W \in \mathcal{W}_b\}$ is a partition of $V(G) - (C \cup K_5)$, where $\mathcal{W}_b = \{\{c_2, c_3\}\} \cup \{W \subseteq \{c_1, c_4, c'_3\} \mid |W| \geq 2\}$.

(3) When $K_5 \neq K_6$ and $N_G(K_5) \cap N_6 \neq \emptyset$,

(3.1) the layout of $G[U]$ in Figure 5.3.7(a) is correct iff $c'_3 \notin N_G(K_5 - \{c_5\})$, $c'_2 \notin N_G(K_6 - \{c_6\})$, and the family $\mathcal{F}_{a'} = \{\mathcal{C}^G_{C,F_4}(\{c_2, c_3\}), \mathcal{C}^G_{C,F_4}(\{c'_2, c_4\}), \mathcal{C}^G_{C,F_4}(\{c'_3, c_1\})\}$ is a partition of $V(G) - (C \cup K_5 \cup K_6)$, and

(3.2) the layout of $G[U]$ in Figure 5.3.7(b) is correct iff $c'_3 \notin N_G(K_5 - \{c_5\})$, $c'_3 \notin N_G(K_6 - \{c_6\})$, $|N_G(K_5) \cap K_6| = |K_5 \cap N_G(K_6)| = 1$, $\{c_1, c_4\} \cup (N_G(K_5) \cap K_6) \cup (K_5 \cap N_G(K_6))$ is an MC_4 in G, and the family $\mathcal{F}_{b'} = \{\mathcal{C}^G_{C,F_4}(W) \mid W \in \mathcal{W}_{b'}\}$ is a partition of $V(G) - (C \cup K_5 \cup K_6)$, where $\mathcal{W}_{b'} = \{\{c_2, c_3\}\} \cup \{W \subseteq \{c_1, c_4, c'_3\} \mid |W| \geq 2\}$.

Proof. Let $\mathcal{L}_a$ (or $\mathcal{L}_b$, respectively) be the layout of $G[U]$ in Figure 5.3.7(a) (respectively, Figure 5.3.7(b)). Let $\mathcal{L}_{a'}$ (or $\mathcal{L}_{b'}$, respectively) be the layout of $G[U]$ obtained from $\mathcal{L}_a$ (respectively, $\mathcal{L}_b$) by exchanging the positions of c'_2 and c'_3.

(1) If $\mathcal{L}_a$ is correct, then it is clear that (1.1), (1.2), and (1.3) hold. Suppose that $N_G(K_5) \cap K_6 = \emptyset$ and (1.1) through (1.3) hold. It suffices to prove that if $\mathcal{L}_{a'}$ is correct, so is $\mathcal{L}_a$. So, assume that $\mathcal{L}_{a'}$ is correct. Then, $\mathcal{C}^G_{C,F_4}(\{c_1, c_4, c'_3\}) = \emptyset$ or else $N_G(K_5) \cap \mathcal{C}^G_{C,F_4}(\{c_1, c_4, c'_3\})$ would be nonempty, contradicting (1.3). Similarly, $\mathcal{C}^G_{C,F_4}(\{c_1, c_4, c'_2\}) = \emptyset$. Now, it is easy to see that $\mathcal{L}_{a'}$ can be transformed to $\mathcal{L}_a$.

(2) Easy.

(3) Assume that $K_5 \neq K_6$ and $N_G(K_5) \cap N_6 \neq \emptyset$. It is easy to see that the necessary conditions in (3.1) and (3.2) hold. First, we show the sufficient condition in (3.1). Suppose that $c'_3 \notin N_G(K_5 - \{c_5\})$, $c'_2 \notin N_G(K_6 - \{c_6\})$, and the family $\mathcal{F}_{a'}$ is a partition of $V(G) - (C \cup K_5 \cup K_6)$. It suffices to prove that each of $\mathcal{L}_{a'}$, $\mathcal{L}_b$, and $\mathcal{L}_{b'}$ can be transformed to $\mathcal{L}_a$. To do this, we need to consider only the following three cases.

Case 1: $\mathcal{L}_{a'}$ is correct. Then, G has a correct embedding as shown in Figure 5.3.8(a). From this embedding, it is easy to see that $\mathcal{L}_a$ is correct.

Case 2: $\mathcal{L}_b$ is correct. Then, G has a correct embedding as shown in Figure 5.3.8(b). From this embedding, it is easy to see that $\mathcal{L}_a$ is correct.

Case 3: $\mathcal{L}_{b'}$ is correct. Then, G has a correct embedding as shown in Figure 5.3.8(c). From this embedding, it is easy to see that $\mathcal{L}_a$ is correct.

Next, we show the sufficient condition in (3.2). Suppose that $c'_3 \notin N_G(K_5 - \{c_5\})$, $c'_3 \notin N_G(K_6 - \{c_6\})$, $|N_G(K_5) \cap K_6| = |K_5 \cap N_G(K_6)| = 1$, $\{c_1, c_4\} \cup (N_G(K_5) \cap K_6) \cup (K_5 \cap N_G(K_6))$ is an MC_4 in G, and the family $\mathcal{F}_{b'}$ is a partition of $V(G) - (C \cup K_5 \cup K_6)$. It suffices to prove that each of $\mathcal{L}_a$, $\mathcal{L}_{a'}$, and $\mathcal{L}_{b'}$ can be transformed to $\mathcal{L}_b$. To do this, we need to consider only the following three cases.

Case 4: $\mathcal{L}_a$ is correct. Then, G has a correct embedding as shown in Figure 5.3.8(d). From this embedding, it is easy to see that $\mathcal{L}_b$ is correct.

Case 5: $\mathcal{L}_{a'}$ is correct. Then, G has a correct embedding as shown in Figure 5.3.8(e). From this embedding, it is easy to see that $\mathcal{L}_b$ is correct.

Case 6: $\mathcal{L}_b$ is correct. Then, G has a correct embedding as shown in Figure 5.3.8(f). From this embedding, it is easy to see that $\mathcal{L}_b$ is correct. ∎

By Lemma 5.5, we can actually compute a layout of $G[U]$ of one of the two forms in Figure 5.3.7 which is correct whenever $G[C]$ has a correct nonpizza layout. So, in the sequel, we may assume that $c'_2 = c_2$ and $c'_3 = c_3$ in Figure 5.3.7.

Theorem 5.6 Assume that the layout of $G[U]$ in Figure 5.3.7(a) is correct whenever $G[C]$ has a correct nonpizza layout. Let F' be the set of the edges $\{d_1, d_2\}$ in $G - U$ such that $\{d_1, d_2, c_1, c_4\}$ is an MC_4 in G. Then, the layout of $G[U]$ in Figure 5.3.7(a) is correct iff $\mathcal{C}^G_{U,F'}(W) = \emptyset$ for every subset W of U such that $W = \{c_1, c_2\}$, $W = \{c_1, c_2, c_5\}$, $W = \{c_3, c_4\}$, $W = \{c_3, c_4, c_6\}$, $\{c_2, c_6\} \subseteq W$, $\{c_3, c_5\} \subseteq W$, or $\{c_5, c_6\} \subseteq W$.

Proof. The necessary condition simply follows from Figure 5.3.7(a) and the fact that the best nonpizza layout of $G[C]$ is of 2-type. To prove the sufficient condition, suppose that $\mathcal{F}$ is a partition of $V(G) - U$. Then, it is not difficult to see that every correct layout of $G[U]$ in which $G[C]$ is layouted as a pizza can be transformed to the one in Figure 5.3.9 which is explicit, and the latter can be transformed to the one in Figure 5.3.7(a). ∎

Now, suppose that the layout of $G[U]$ in Figure 5.3.7(b) is correct whenever $G[C]$ has a correct nonpizza layout. Recall that $c'_2 = c_2$ and $c'_3 = c_3$ in the Figure 5.3.7(b). In the remainder of this section, we show how to decide whether or not this layout of $G[U]$ is really correct. Let $\mathcal{L}_{np}$ be the layout of $G[U]$ in Figure 5.3.7(b). The following (A_1) is a necessary condition for $\mathcal{L}_{np}$ to be correct:

(A_1) $\mathcal{C}^G_{U,F}(\{c_1, c_2\}) = \mathcal{C}^G_{U,F}(\{c_2, c_4\}) = \emptyset$ and $\mathcal{C}^G_{U,F}(\{c_3, c_5\} \cup W) = \mathcal{C}^G_{U,F}(\{c_3, c_6\} \cup W) = \emptyset$ for every $W \subset U$.

Thus, we may assume that (A_1) holds; otherwise, $G[C]$ has no correct nonpizza layout. By this assumption, it is easy to see that every correct layout of $G[U]$ in which $G[C]$ is layouted as a pizza can be transformed to one of the four layouts in Figure 5.3.10. All the four figures in Figure 5.3.10 are explicit.

Similarly to Lemma 4.9, we can prove the following fact.

Fact 7 Let $b_1 = 1$ if either there is no $d \notin U$ such that $\{d, c_1, c_2, c_5\}$ is an MC_4 in G or the unique $d \notin U$ such that $\{d, c_1, c_2, c_5\}$ is an MC_4 in G belongs to $\mathcal{C}^G_{U,F}(\{c_1, c_2, c_5\})$; otherwise, let $b_1 = 0$. Let $b_2 = 1$ if either there is no $d \notin U$ such that $\{d, c_2, c_4, c_6\}$ is an MC_4 in G or the unique $d \notin U$ such that $\{d, c_2, c_4, c_6\}$ is an MC_4 in G belongs to $\mathcal{C}^G_{U,F}(\{c_2, c_4, c_6\})$; otherwise, let $b_2 = 0$. Then, the following (1) through (4) hold:

(1) If $b_1 = 1$ and $b_2 = 1$, then every correct layout of $G[U]$ in which $G[C]$ is layouted as a pizza can be transformed to the layout in Figure 5.3.10(a).

(2) If $b_1 = 0$ and $b_2 = 1$, then every correct layout of $G[U]$ in which $G[C]$ is layouted as a pizza can be transformed to the layout in Figure 5.3.10(b).

(3) If $b_1 = 1$ and $b_2 = 0$, then every correct layout of $G[U]$ in which $G[C]$ is layouted as a pizza can be transformed to the layout in Figure 5.3.10(c).

(4) If $b_1 = 0$ and $b_2 = 0$, then every correct layout of $G[U]$ in which $G[C]$ is layouted as a pizza can be transformed to the layout in Figure 5.3.10(d).

Lemma 5.7 Suppose that there is a country $c_7 \notin U$ such that both $\{c_1, c_7\} \in E(G)$ and $\{c_4, c_7\} \in E(G)$. Then, we can decide in polynomial time whether $\mathcal{L}_{np}$ is really correct.
Proof. Let $U_7 = U \cup \{c_7\}$. Then, every correct layout of $G[U_7]$ in which $G[C]$ is layouted as a pizza can be transformed to the one in Figure 5.3.10+. This figure is explicit. We may assume that $\{c_7, c_2\}$, $\{c_7, c_5\}$, and $\{c_7, c_6\}$ are edges in G and that there is no $d \notin U_7$

such that $\{c_1, c_4, c_7, d\}$ is an MC$_4$ in G; otherwise, the layout in Figure 5.3.10+ is incorrect. Then, $C_5 = \{c_1, c_2, c_4, c_7\}$ is an MC$_4$ in G. We claim that $\mathcal{L}_{np}$ is incorrect. Assume, on the contrary, that $\mathcal{L}_{np}$ is correct. Then, $G[C_5]$ has a correct nonpizza layout. So, the best nonpizza layout of $G[C_5]$ is not the rice-ball by our assumption. Thus, by $\mathcal{L}_{np}$, there is some $d \notin U_7$ such that $\{c_1, c_4, c_7, d\}$ is an MC$_4$ in G. This is a contradiction. ∎

By Lemma 5.7, we may assume that the following (A_2) holds:
(A_2) There is no country $c_7 \notin U$ such that both $\{c_1, c_7\} \in E(G)$ and $\{c_4, c_7\} \in E(G)$.

Lemma 5.8 If $\mathcal{C}^G_{U,F}(\{c_2, c_3\}) \neq \emptyset$ or $\mathcal{C}^G_{U,F}(\{c_5, c_6\}) \neq \emptyset$, then we can decide in polynomial time whether $\mathcal{L}_{np}$ is really correct.
Proof. If $\mathcal{C}^G_{U,F}(\{c_2, c_3\}) \neq \emptyset$, then by (A_1), each of the four layouts in Figure 5.3.10 can be transformed to $\mathcal{L}_{np}$. Suppose that $\mathcal{C}^G_{U,F}(\{c_2, c_3\}) = \emptyset$ but $\mathcal{C}^G_{U,F}(\{c_5, c_6\}) \neq \emptyset$. Then, we claim that $\mathcal{L}_{np}$ is correct iff the family $\mathcal{F} = \{\mathcal{C}^G_{U,F}(W) \mid W \in \mathcal{W}\}$ is a partition of $V(G) - U$, where $\mathcal{W}$ is the family of the following sets: $\{c_1, c_3\}$, …, $\{c_1, c_6\}$, $\{c_2, c_5\}$, $\{c_2, c_6\}$, $\{c_3, c_4\}$, $\{c_4, c_5\}$, $\{c_4, c_6\}$, $\{c_5, c_6\}$, $\{c_1, c_3, c_4\}$, $\{c_1, c_4, c_5\}$, $\{c_2, c_5, c_6\}$, $\{c_1, c_5, c_6\}$, $\{c_4, c_5, c_6\}$, and $\{c_1, c_4, c_5, c_6\}$. The necessary condition of this claim is obvious. To show the sufficient condition, suppose that $\mathcal{F}$ is a partition of $V(G) - U$. Then by Figure 5.3.10, it is clear that every correct of $G[U]$ in which $G[C]$ is layouted as a pizza can be transformed to the layout in Figure 5.3.11 which is explicit. Obviously, the latter layout can be transformed to $\mathcal{L}_{np}$. ∎

By Lemma 5.8, we may assume that $\mathcal{C}^G_{U,F}(\{c_2, c_3\}) = \mathcal{C}^G_{U,F}(\{c_5, c_6\}) = \emptyset$. By this assumption, (A_1), and Figure 5.3.10, a necessary condition for $G[C]$ to have a correct pizza layout is that in the graph $G - \mathcal{C}^G_U(\{c_1, c_3\}) - \mathcal{C}^G_U(\{c_3, c_4\})$, the neighbors of c_3 are only c_1, c_2, c_4, c_5, and c_6. Also note that the countries in $\mathcal{C}^G_U(\{c_1, c_3\}) \cup \mathcal{C}^G_U(\{c_3, c_4\})$ have no effect on distinguishing $\mathcal{L}_{np}$ with the four layouts in Figure 5.3.10. Thus, we may assume that the following (A_3) holds:
(A_3) $N_G(c_3) = \{c_1, c_2, c_4, c_5, c_6\}$.

Lemma 5.9 Assume that every correct layout of $G[U]$ in which $G[C]$ is layouted as a pizza can be transformed to the layout in Figure 5.3.10(a). Let $X = \mathcal{C}^G_{U,F}(\{c_1, c_2, c_5\})$, $Y = \mathcal{C}^G_{U,F}(\{c_2, c_5, c_6\})$, and $Z = \mathcal{C}^G_{U,F}(\{c_2, c_4, c_6\})$. Then, $\mathcal{L}_{np}$ is correct iff the following (1) and (2) hold:

(1) If $X \neq \emptyset$ and $Y \neq \emptyset$, then $|N_G(X) \cap Y| = |X \cap N_G(Y)| = 1$, $\{c_2, c_5\} \cup (N_G(X) \cap Y) \cup (X \cap N_G(Y))$ is an MC$_4$ in G, $X \cap N_G(c_2) = X \cap N_G(Y)$, and $N_G(c_5) \cap Y = N_G(X) \cap Y$.

(2) If $Y \neq \emptyset$ and $Z \neq \emptyset$, then $|N_G(Z) \cap Y| = |Z \cap N_G(Y)| = 1$, $\{c_2, c_6\} \cup (N_G(Z) \cap Y) \cup (Z \cap N_G(Y))$ is an MC$_4$ in G, $Z \cap N_G(c_2) = Z \cap N_G(Y)$, and $N_G(c_6) \cap Y = N_G(Z) \cap Y$.

Proof. The necessary condition is obvious. Let $\mathcal{L}_{p,a}$ be the layout of $G[U]$ in Figure 5.3.10(a). To prove the sufficient condition, it suffices to prove that if (1) and (2) hold, $\mathcal{L}_{p,a}$ can be transformed to $\mathcal{L}_{np}$. So, suppose that $\mathcal{L}_{p,a}$ is correct and (1) and (2) hold. Then, by the assumption we have made so far, the family $\mathcal{F} = \{\mathcal{C}^G_{U,F}(W) \mid W \in \mathcal{W}\}$ is a partition of $V(G) - U$, where $\mathcal{W}$ is the family of the following sets: $\{c_1, c_5\}$, $\{c_2, c_5\}$, $\{c_2, c_6\}$, $\{c_4, c_6\}$, $\{c_1, c_2, c_5\}$, $\{c_2, c_4, c_6\}$, and $\{c_2, c_5, c_6\}$. Consider the following three cases:

Case 1: $\mathcal{C}^G_{U,F}(\{c_2, c_5, c_6\}) = \emptyset$. Then, it is clear that $\mathcal{L}_{p,a}$ can be transformed to $\mathcal{L}_{np}$.

Case 2: $\mathcal{C}^G_{U,F}(\{c_2, c_5, c_6\}) \neq \emptyset$ and $\mathcal{C}^G_{U,F}(\{c_1, c_2, c_5\}) \neq \emptyset$ but $\mathcal{C}^G_{U,F}(\{c_2, c_4, c_6\}) = \emptyset$. Let $x = X \cap N_G(Y)$ and $y = N_G(X) \cap Y$. Since (1) and (2) hold, the embedding of G in Figure 5.3.12(a) is correct. ¿From this embedding, it is clear that $\mathcal{L}_{np}$ is correct.

Case 3: $\mathcal{C}^G_{U,F}(\{c_2, c_5, c_6\}) \neq \emptyset$ and $\mathcal{C}^G_{U,F}(\{c_2, c_4, c_6\}) \neq \emptyset$ but $\mathcal{C}^G_{U,F}(\{c_1, c_2, c_5\}) = \emptyset$. Similar to Case 2.

Case 4: $\mathcal{C}^G_{U,F}(\{c_2, c_5, c_6\}) \neq \emptyset$, $\mathcal{C}^G_{U,F}(\{c_2, c_4, c_6\}) \neq \emptyset$, and $\mathcal{C}^G_{U,F}(\{c_1, c_2, c_5\}) \neq \emptyset$. Let $x = X \cap N_G(Y)$, $y_1 = N_G(X) \cap Y$, $z = Z \cap N_G(Y)$, and $y_2 = N_G(Z) \cap Y$. Since (1) and (2) hold, the embedding of G in Figure 5.3.12(b) is correct. ¿From this embedding, it is clear that $\mathcal{L}_{np}$ is correct. ∎

Lemma 5.10 Assume that every correct layout of $G[U]$ in which $G[C]$ is layouted as a pizza can be transformed to the layout in Figure 5.3.10(b). Then, we can decide in polynomial time whether $\mathcal{L}_{np}$ is really correct.

Proof. Let $\mathcal{L}_{p,b}$ be the layout of $G[U]$ in Figure 5.3.10(b). By Fact 7 and Lemma 5.9, we may assume that there is a country $c_7 \notin U$ such that $\{c_1, c_2, c_5, c_7\}$ is an MC$_4$ in G and $c_7 \notin \mathcal{C}^G_{U,F}(\{c_1, c_2, c_5\})$. The following (B_1), (B_2), and (B_3) are necessary conditions for $\mathcal{L}_{p,b}$ to be correct:

(B_1) The family $\{\mathcal{C}^G_{U,F}(W) \mid W \in \mathcal{W}\}$ is a partition of $V(G) - U$, where $\mathcal{W}$ is the family of the following sets: $\{c_1, c_5\}$, $\{c_2, c_6\}$, $\{c_4, c_6\}$, $\{c_2, c_4, c_6\}$, $\{c_2, c_5, c_6\}$, and $\{c_1, c_2, c_5, c_6\}$. Note that $\mathcal{C}^G_{U,F}(\{c_2, c_5\}) = \emptyset$ or else c_7 would be in $\mathcal{C}^G_{U,F}(\{c_1, c_2, c_5\})$ by Figure 5.3.10(b).

(B_2) $N_G(c_1) \cap \mathcal{C}^G_{U,F}(\{c_1, c_2, c_5, c_6\}) = \{c_7\}$.

(B_3) $\mathcal{C}^G_{U,F}(\{c_1, c_5\}) = \mathcal{C}^G_U(\{c_1, c_5\})$.

We may assume that G satisfies the conditions (B_1), (B_2), and (B_3) hold; otherwise, $\mathcal{L}_{p,b}$ is incorrect and hence $\mathcal{L}_{np}$ must be correct. A simple but useful consequence of (B_1) is that no country can be adjacent to both c_4 and c_5 in G. By (B_1) and (B_2), a necessary condition for $\mathcal{L}_{p,b}$ to be correct is that in the graph $G - \mathcal{C}^G_U(\{c_1, c_5\})$, the neighbors of c_1 are only c_2, …, c_5, and c_7. Also note that the countries in $\mathcal{C}^G_U(\{c_1, c_5\})$ have no effect on distinguishing $\mathcal{L}_{np}$ and $\mathcal{L}_{p,b}$. Thus, we may assume that the following (B_4) holds:

(B_4) $N_G(c_1) = \{c_2, \ldots, c_5, c_7\}$.

Let $U_7 = U \cup \{c_7\}$, and F_7 be the set of the edges $\{d_1, d_2\}$ in $G - U_7$ such that $\{d_1, d_2, c_i, c_j\}$ is an MC$_4$ in G for some $c_i \in U_7$ and $c_j \in U_7$. Consider the following three cases:

Case 1: $\{c_6, c_7\} \in E(G)$, and either there is no $d \notin U_7$ such that $\{c_2, c_6, c_7, d\}$ is an MC$_4$ in G or the unique $d \notin U_7$ such that $\{c_2, c_6, c_7, d\}$ is an MC$_4$ in G belongs to $\mathcal{C}^G_{U_7,F_7}(\{c_2, c_6, c_7\})$. Then, every correct layout of $G[U_7]$ in which $G[C]$ is layouted as a pizza can be transformed to the one in Figure 5.3.13(a). Figure 5.3.13(a) is explicit. If this layout is correct, then the family $\mathcal{F} = \{\mathcal{C}^G_{U_7,F_7}(W) \mid W \in \mathcal{W}\}$ is a partition of $V(G) - U_7$, where $\mathcal{W}$ is the family of the following sets: $\{c_2, c_6\}$, $\{c_2, c_7\}$, $\{c_4, c_6\}$, $\{c_5, c_6\}$, $\{c_5, c_7\}$, $\{c_6, c_7\}$, $\{c_2, c_4, c_6\}$, $\{c_2, c_6, c_7\}$, and $\{c_5, c_6, c_7\}$. We may assume that $\mathcal{F}$ is really a partition of $V(G) - U_7$; otherwise, $\mathcal{L}_{np}$ must be correct. Then, every correct layout of $G[U_7]$ in which $G[C]$ is layouted as a nonpizza can be transformed to the one in Figure 5.3.13(b). Figure 5.3.13(b) is explicit. Clearly, the layout in Figure 5.3.13(a) can be transformed to the layout in Figure 5.3.13(b) iff the three sets $\mathcal{C}^G_{U_7,F_7}(\{c_5, c_6\})$, $\mathcal{C}^G_{U_7,F_7}(\{c_2, c_4, c_6\})$, and $\mathcal{C}^G_{U_7,F_7}(\{c_5, c_6, c_7\})$ are all empty.

Case 2: $\{c_6, c_7\} \in E(G)$ and the unique $c_8 \notin U_7$ such that $\{c_2, c_6, c_7, c_8\}$ is an MC$_4$ in G does not belong to $\mathcal{C}^G_{U_7,F_7}(\{c_2, c_6, c_7\})$. Then, by (A_3), (B_1) and (B_4), either $c_8 \in$

$\mathcal{C}^G_{U_7,F_7}(\{c_2,c_6,c_7,c_5\})$ or $c_8 \in \mathcal{C}^G_{U_7,F_7}(\{c_2,c_6,c_7,c_4\})$. Let $U_8 = \{c_1,\dots,c_8\}$.

Case 2.1: $c_8 \in \mathcal{C}^G_{U_7,F_7}(\{c_2,c_6,c_7,c_5\})$. Then, every correct layout of $G[U_8]$ in which $G[C]$ is layouted as a pizza can be transformed to the one in Figure 5.3.14(a). Figure 5.3.14(a) is explicit except that possibly, c_8 may meet c_5. By Figure 5.3.14(a), we may assume that $\mathcal{C}^G_{U_8}(\{c_2,c_5\}) = \emptyset$ and there is no $d \notin U_8$ such that $\{c_2,c_5,c_7,d\}$ is an MC_4 in G; otherwise, the layout in Figure 5.3.14(a) is incorrect. Then, every correct layout of $G[U_8]$ in which $G[C]$ is layouted as a nonpizza can be transformed to the one in Figure 5.3.14(b). Figure 5.3.14(b) is explicit except that possibly, c_8 may meet c_5. Let K_8 be the connected component in $G - U_7$ containing c_8. Since $c_8 \in \mathcal{C}^G_{U_7,F_7}(\{c_2,c_6,c_7,c_5\})$, $c_5 \in N_G(K_8)$. Now, it is clear that the layout of $G[U_8]$ in Figure 5.3.14(b) is correct iff $N_G(c_6) \cap K_8 = \{c_8\}$ and the family $\{\mathcal{C}^G_{U_7}(W) \mid W \in \mathcal{W}\}$ is a partition of $V(G) - (U_7 \cup K_8)$, where $\mathcal{W}$ is the family of the following sets: $\{c_4,c_6\}$, $\{c_4,c_7\}$, $\{c_5,c_7\}$, $\{c_6,c_7\}$, and $\{c_4,c_6,c_7\}$.

Case 2.2: $c_8 \in \mathcal{C}^G_{U_7,F_7}(\{c_2,c_6,c_7,c_4\})$. Then, every correct layout of $G[U_8]$ in which $G[C]$ is layouted as a pizza can be transformed to the one in Figure 5.3.15(a). Figure 5.3.15(a) is explicit except that possibly, c_8 may meet c_4. By Figure 5.3.15(a), we may assume that $\mathcal{C}^G_{U_8}(\{c_2,c_5\}) = \emptyset$, $\mathcal{C}^G_{U_8}(\{c_2,c_5,c_7\}) = \emptyset$, and there is no $d \notin U_8$ such that $\{c_2,c_5,c_7,d\}$ is an MC_4 in G; otherwise, the layout in Figure 5.3.15(a) is incorrect. Then, every correct layout of $G[U_8]$ in which $G[C]$ is layouted as a nonpizza can be transformed to the one in Figure 5.3.15(b). Figure 5.3.15(b) is explicit except that possibly, c_8 may meet c_4. Let K_8 be the connected component in $G - U_7$ containing c_8. Since $c_8 \in \mathcal{C}^G_{U_7,F_7}(\{c_2,c_6,c_7,c_4\})$, $c_4 \in N_G(K_8)$. Now, it is clear that the layout of $G[U_8]$ in Figure 5.3.14(b) is correct iff $N_G(c_2) \cap K_8 = \{c_8\}$ and the family $\{\mathcal{C}^G_{U_7}(W) \mid W \in \mathcal{W}\}$ is a partition of $V(G) - (U_7 \cup K_8)$, where $\mathcal{W}$ is the family of the following sets: $\{c_2,c_6\}$, $\{c_2,c_7\}$, $\{c_4,c_6\}$, $\{c_4,c_7\}$, and $\{c_5,c_7\}$.

Case 3: $\{c_6,c_7\} \notin E(G)$. Then, every correct layout of $G[U_7]$ in which $G[C]$ is layouted as a pizza (or nonpizza, respectively) can be transformed to the one in Figure 5.3.16(a) (respectively, the one in Figure 5.3.16(b) or the one in Figure 5.3.16(c)). All the three figures are explicit. We assume that the following (B_5) holds:

(B_5) $\mathcal{C}^G_{U_7,F_7}(\{c_5,c_6\}) = \mathcal{C}^G_{U_7,F_7}(\{c_2,c_5,c_6\}) = \mathcal{C}^G_{U_7,F_7}(\{c_5,c_6,c_7\}) = \emptyset$.

If (B_5) does not hold, then the last two layouts in Figure 5.3.16 are incorrect and $\mathcal{L}_{np}$ is incorrect. By (B_5), if $\mathcal{C}^G_{U_7,F_7}(\{c_2,c_5\}) \neq \emptyset$, then the layout of $G[U_7]$ in Figure 5.3.16(a) can be transformed to the one in Figure 5.3.16(b) and hence $\mathcal{L}_{np}$ is correct. So, we may further assume that the following (B_6) holds:

(B_6) $\mathcal{C}^G_{U_7,F_7}(\{c_2,c_5\}) = \emptyset$.

Case 3.1: $\mathcal{C}^G_{U_7,F_7}(\{c_6,c_7\}) \neq \emptyset$. Then, we may assume that there is no $d \notin U_7$ such that $\{c_2,c_5,c_7,d\}$ is an MC_4 in G; otherwise, C has no correct pizza layout and $\mathcal{L}_{np}$ must be correct. By this assumption, the layout of $G[U_7]$ in Figure 5.3.16(c) can be transformed to the layout in Figure 5.3.16(b). Moreover, the layout of $G[U_7]$ in Figure 5.3.16(a) can be transformed to the layout in Figure 5.3.16(b) iff $\mathcal{C}^G_{U_7,F_7}(\{c_2,c_4,c_6\}) = \mathcal{C}^G_{U_7,F_7}(\{c_5,c_6,c_7) = \emptyset$.

Case 3.2: Either there is no $d \notin U_7$ such that $\{c_2,c_5,c_7,d\}$ is an MC_4 in G or the unique $d \notin U_7$ such that $\{c_2,c_5,c_7,d\}$ is an MC_4 in G belongs to $\mathcal{C}^G_{U_7,F_7}(\{c_2,c_5,c_7\})$. Then, every correct layout of $G[U_7]$ in which $G[C]$ is layouted as a nonpizza can be transformed to the one in Figure 5.3.16(b). By Case 3.1, we may assume that $\mathcal{C}^G_{U_7,F_7}(\{c_6,c_7\}) = \emptyset$. Let $X = \mathcal{C}^G_{U_7,F_7}(\{c_2,c_5,c_7\})$, $Y = \mathcal{C}^G_{U_7,F_7}(\{c_2,c_6,c_7\})$, and $Z = \mathcal{C}^G_{U_7,F_7}(\{c_2,c_4,c_6\})$. Then, it is not difficult to see that the layout of $G[U_7]$ in Figure 5.3.16(b) is correct iff the following (i), (ii), and (iii) hold:

(i) $\mathcal{C}^G_{U_7,F_7}(\{c_5, c_6\} \cup W) = \emptyset$ for every $W \subset U_7$.

(ii) If $N_G(X) \cap Y \neq \emptyset$, then $\mathcal{C}^G_{U_7,F_7}(\{c_2, c_7\}) = \emptyset$.

(iii) If both Y and Z are nonempty, then $|N_G(Y) \cap Z| = |Y \cap N_G(Z)| = 1$, $\{c_2, c_6\} \cup (N_G(Y) \cap Z) \cup (Y \cap N_G(Z))$ is an MC_4 in G, $N_G(c_2) \cap Z = N_G(Y) \cap Z$, and $Y \cap N_G(c_6) = Y \cap N_G(Z)$.

Case 3.3: $\{c_6, c_7\} \notin E(G)$ and the unique $c_8 \notin U_7$ such that $\{c_2, c_5, c_7, c_8\}$ is an MC_4 in G does not belong to $\mathcal{C}^G_{U_7,F_7}(\{c_2, c_5, c_7\})$. Then, every correct layout of $G[U_7]$ in which $G[C]$ is layouted as a nonpizza can be transformed to the one in Figure 5.3.16(c). If the layout in Figure 5.3.16(c) is really correct, then $\mathcal{C}^G_{U_7,F_7}(\{c_2, c_5\}) = \mathcal{C}^G_{U_7}(\{c_2, c_5\})$ and hence by (B_6), the neighbors of c_5 in the graph $G - \mathcal{C}^G_{U_7}(\{c_5, c_7\})$ are only c_1, c_2, c_3, c_7, and c_8. Also note that the countries in $\mathcal{C}^G_{U_7}(\{c_5, c_7\})$ have no effect on distinguishing the two layouts in Figure 5.3.16(a) and Figure 5.3.16(c). Thus, by (B_6), we may assume that the following (B_7) holds:

(B_7) $N_G(c_5) = \{c_1, c_2, c_3, c_7, c_8\}$.

Let $U_8 = U_7 \cup \{c_8\}$, and F_8 be the set of the edges $\{d_1, d_2\}$ in $G - U_8$ such that $\{d_1, d_2, c_i, c_j\}$ is an MC_4 in G for some $c_i \in U_8$ and $c_j \in U_8$. By (B_1), $\{c_8, c_4\} \notin E(G)$. We distinguish three cases as follows.

Case 3.3.1: $\{c_8, c_6\} \in E(G)$, and either there is no $d \notin U_8$ such that $\{c_2, c_6, c_8, d\}$ is an MC_4 in G or the unique $d \notin U_8$ such that $\{c_2, c_6, c_8, d\}$ is an MC_4 in G belongs to $\mathcal{C}^G_{U_8,F_8}(\{c_2, c_6, c_8\})$. Then, by (B_7), every correct layout of $G[U_8]$ in which $G[C]$ is layouted as a nonpizza can be transformed to the one in Figure 5.3.17(b). Figure 5.3.17(b) is explicit. We may assume that there is no $d \notin U_8$ such that $\{c_2, c_7, c_8, d\}$ is an MC_4 in G; otherwise, the layout in Figure 5.3.17(b) is incorrect. Then, every correct layout of $G[U_8]$ in which $G[C]$ is layouted as a pizza can be transformed to the one in Figure 5.3.17(a). Figure 5.3.17(a) is explicit. Clearly, the layout in Figure 5.3.17(b) is really correct iff $\mathcal{C}^G_{U_8,F_8}(\{c_2, c_4, c_6\}) = \emptyset$ and $\mathcal{C}^G_{U_8,F_8}(\{c_2, c_7, c_8\}) = \emptyset$.

Case 3.3.2: $\{c_8, c_6\} \in E(G)$, and the unique $c_9 \notin U_8$ such that $\{c_2, c_6, c_8, c_9\}$ is an MC_4 in G does not belong to $\mathcal{C}^G_{U_8,F_8}(\{c_2, c_6, c_8\})$. Let $U_9 = U_8 \cup \{c_9\}$, and F_9 be the set of the edges $\{d_1, d_2\}$ in $G - U_9$ such that $\{d_1, d_2, c_i, c_j\}$ is an MC_4 in G for some $c_i \in U_9$ and $c_j \in U_9$. We may assume that $\{c_9, c_4\} \notin E(G)$ or $\{c_9, c_7\} \notin E(G)$; otherwise, $\mathcal{L}_{p,b}$ is incorrect.

Case 3.3.2.1: $\{c_9, c_4\} \in E(G)$. Then, by (B_7), every correct layout of $G[U_9]$ in which $G[C]$ is layouted as a pizza can be transformed to the one in Figure 5.3.18(a). Figure 5.3.18(a) becomes explicit after c_9 is deleted from it. The following (B'_1) and (B'_2) are two necessary conditions for the layout in Figure 5.3.18(a) to be correct.

(B'_1) $\{c_9, c_7\} \notin E(G)$.

(B'_2) There is no $d \notin U_9$ such that $\{c_4, c_6, c_9, d\}$ is an MC_4 in G, $d \notin \mathcal{C}^G_{U_9,F_9}(\{c_4, c_6, c_9\})$, and $d \notin \mathcal{C}^G_{U_9,F_9}(\{c_2, c_4, c_6, c_9\})$.

We may assume that both (B'_1) and (B'_2) hold; otherwise, $\mathcal{L}_{np}$ must be correct. Then, by (B_7), every correct layout of $G[U_9]$ in which $G[C]$ is layouted as a nonpizza can be transformed to the one in Figure 5.3.18(b). Figure 5.3.18(b) is explicit. The following (B'_3) and (B'_4) are two necessary conditions for the layout in Figure 5.3.18(b) to be correct.

(B'_3) There is no $d \notin U_9$ such that $\{c_2, c_4, c_9, d\}$ is an MC_4 in G.

(B'_4) There is no $d \notin U_9$ such that $\{c_4, c_6, c_9, d\}$ is an MC_4 in G and $d \notin \mathcal{C}^G_{U_9,F_9}(\{c_4, c_6, c_9\})$.

We may assume that both (B'_3) and (B'_4) hold; otherwise, $\mathcal{L}_{np}$ is incorrect. Then, we may assume that Figure 5.3.18(a) is explicit. Now, it is easy to see that the layout in Figure 5.3.18(b) is correct iff $\mathcal{C}^G_{U_9,F_9}(W) = \emptyset$ for each subset W of U_9 such that $\{c_2, c_7\} \subseteq W$, $\{c_2, c_9\} \subseteq W$, or $\{c_6, c_8\} \subseteq W$.

Case 3.3.2.2: $\{c_9, c_4\} \notin E(G)$ but $\{c_9, c_7\} \in E(G)$. Then, by (B_7), every correct layout of $G[U_9]$ in which $G[C]$ is layouted as a pizza can be transformed to the one in Figure 5.3.18-(a) or the one in Figure 5.3.18-(b). Both Figure 5.3.18-(a) and Figure 5.3.18-(b) become explicit after c_9 is deleted from them. The following (B'_5) is a necessary condition for each of the two layouts in Figure 5.3.18-(a) and Figure 5.3.18-(b) to be correct.

(B'_5) There is no $d \notin U_9$ such that $\{c_7, c_8, c_9, d\}$ is an MC_4 in G, $d \notin \mathcal{C}^G_{U_9,F_9}(\{c_7, c_8, c_9\})$, and $d \notin \mathcal{C}^G_{U_9,F_9}(\{c_2, c_7, c_8, c_9\})$.

We may assume that (B'_5) hold; otherwise, $\mathcal{L}_{np}$ must be correct. Then, by (B_7), every correct layout of $G[U_9]$ in which $G[C]$ is layouted as a nonpizza can be transformed to the one in Figure 5.3.18-(c). Figure 5.3.18-(c) is explicit. The following (B'_6) and (B'_7) are two necessary conditions for the layout in Figure 5.3.18-(c) to be correct.

(B'_6) There is no $d \notin U_9$ such that $\{c_6, c_8, c_9, d\}$, $\{c_2, c_6, c_9, d\}$, or $\{c_2, c_7, c_9, d\}$ is an MC_4 in G.

(B'_7) There is no $d \notin U_9$ such that $\{c_7, c_8, c_9, d\}$ is an MC_4 in G and $d \notin \mathcal{C}^G_{U_9,F_9}(\{c_7, c_8, c_9\})$.

We may assume that both (B'_6) and (B'_7) hold; otherwise, $\mathcal{L}_{np}$ is incorrect. Then, we may assume that both Figure 5.3.18-(a) and Figure 5.3.18-(b) are explicit. Now, it is easy to see that the layout in Figure 5.3.18-(c) is correct iff $\mathcal{C}^G_{U_9,F_9}(W) = \emptyset$ for each subset W of U_9 such that $W = \{c_2, c_4, c_6\}$, $\{c_2, c_7\} \subseteq W$, $\{c_2, c_9\} \subseteq W$, or $\{c_6, c_8\} \subseteq W$.

Case 3.3.2.3: $\{c_9, c_4\} \notin E(G)$ and $\{c_9, c_7\} \notin E(G)$. Then, by (B_7), every correct layout of $G[U_9]$ in which $G[C]$ is layouted as a nonpizza (or pizza, respectively) can be transformed to the one in Figure 5.3.18+(c) (respectively, the one in Figure 5.3.18+(a) or the one in Figure 5.3.18+(b)). All the three figures in Figure 5.3.18 are explicit. The following (B'_8) is a necessary condition for the layout of $G[U_9]$ in Figure 5.3.18+(c) to be correct:

(B'_8) $\mathcal{C}^G_{U_9,F_9}(W) = \emptyset$ for every subset W of U_9 such that $\{c_2, c_7\} \subseteq W$, $\{c_2, c_9\} \subseteq W$, or $W = \{c_6, c_8\}$.

We may assume that (B'_8) holds; otherwise, $\mathcal{L}_{np}$ is incorrect. Then, every correct layout of $G[U_9]$ in which $G[C]$ is layouted as a pizza can be transformed to the one in Figure 5.3.18+(a) or the one in Figure 5.3.18+(b). Both Figure 5.3.18+(a) and Figure 5.3.18+(b) are explicit. Let K_9 be the connected component in $G - U_8 - F_8$ containing c_9. Now, it is easy to see that the layout of $G[U_9]$ in Figure 5.3.28(b) is correct iff $N_G(c_2) \cap K_9 = \{c_9\}$.

Case 3.3.3: $\{c_8, c_6\} \notin E(G)$. Then, by (B_7), every correct layout of $G[U_8]$ in which $G[C]$ is layouted as a nonpizza (or pizza, respectively) can be transformed to the one in Figure 5.3.19(c) (respectively, in Figure 5.3.19(a) or in Figure 5.3.19(b)). All the three figures are explicit. If $\mathcal{C}^G_{U_8,F_8}(\{c_6, c_8\}) \neq \emptyset$, then by Figure 5.3.19, it is easy to see that the layout of $G[U_8]$ in Figure 5.3.19(c) is correct iff $\mathcal{C}^G_{U_8,F_8}(W) = \emptyset$ for every $W \subseteq U_8$ such that $\{c_2, c_4\} \subseteq W$ or $\{c_2, c_7\} \subseteq W$. Thus, we may assume that $\mathcal{C}^G_{U_8,F_8}(\{c_6, c_8\}) = \emptyset$. Moreover, if every correct layout of $G[U_8]$ in which $G[C]$ is layouted as a pizza can be transformed to the one in Figure 5.3.19(a), then it is easy to prove a lemma similar to Lemma 5.9. Thus, we can assume that every correct layout of $G[U_8]$ in which $G[C]$ is layouted as a pizza can be transformed to the one in Figure 5.3.19(b) but not to the one in Figure 5.3.19(a). Then, there is a country $c_9 \notin U_8$ such that $\{c_2, c_7, c_8, c_9\}$ is an MC_4 in G and $c_9 \notin \mathcal{C}^G_{U_8,F_8}(\{c_2, c_7, c_8\})$. Let $U_9 = U_8 \cup \{c_9\}$. The following (B'_9) and (B'_{10}) are necessary conditions for the layout in Figure 5.3.19(b) to be correct:

(B'_9) $c_9 \in \mathcal{C}^G_{U_8,F_8}(\{c_2, c_6, c_7, c_8\})$.

(B'_{10}) In the graph $G - (\mathcal{C}^G_{U_8}(\{c_2, c_7\}) \cup \mathcal{C}^G_{U_8}(\{c_7, c_8\}))$, the neighbors of c_7 are only c_1, c_2, c_5, c_8, and c_9.

We may assume that (B'_9) and (B'_{10}) hold; otherwise, the layout in Figure 5.3.19(b) is incorrect and $\mathcal{L}_{np}$ must be correct. By (B'_9), a necessary condition for the layout in Figure 5.3.19(c) to be correct is that $\mathcal{C}^G_{U_8}(\{c_2, c_7\}) = \emptyset$. Also note that the countries in $\mathcal{C}^G_{U_8}(\{c_7, c_8\})$ have no effect on distinguishing the last two layouts in Figure 5.3.19. Thus, we may assume that the following (B'_{11}) holds:

(B'_{11}) $N_G(c_7) = \{c_1, c_2, c_5, c_8, c_9\}$.

Now, we construct a new graph G' from G by deleting c_5 and c_7 and adding the three new edges $\{c_8, c_3\}$, $\{c_8, c_1\}$, and $\{c_9, c_1\}$. Let $U' = \{c_1, \ldots, c_4, c_6, c_8\}$, and F' be the set of the edges $\{d_1, d_2\}$ in $G' - U'$ such that $\{d_1, d_2, c_i, c_j\}$ is an MC_4 in G' for some $c_i \in U'$ and $c_j \in U'$. By (B'_9), $c_9 \notin \mathcal{C}^{G'}_{U',F'}(\{c_1, c_2, c_8\})$. Figure 5.3.20 shows two layouts of $G'[U']$. Both figures in Figure 5.3.20 are explicit. By (B_7), (B'_{11}), and the construction of G', the following (i) and (ii) hold:

(i) The layout of $G[U_8]$ in Figure 5.3.19(b) is correct iff the layout of $G'[U']$ in Figure 5.3.20(a) is correct.

(ii) The layout of $G[U_8]$ in Figure 5.3.19(c) is correct iff the layout of $G'[U']$ in Figure 5.3.20(b) is correct.

Thus, it suffices to distinguish the two layouts in Figure 5.3.20. Note that distinguishing the two layouts in Figure 5.3.20 is a sub-task of distingushing $\mathcal{L}_{p,b}$ and $\mathcal{L}_{np}$, because $\{c_1, c_2, c_8, c_9\}$ is an MC_4 in G' and $c_9 \notin \mathcal{C}^{G'}_{U',F'}(\{c_1, c_2, c_8\})$. Hence, we may recursively distinguish the two layouts in Figure 5.3.20. ∎

Similarly to Lemma 5.10, we can prove the following:

Lemma 5.11 Assume that every correct layout of $G[U]$ in which $G[C]$ is layouted as a pizza can be transformed to the layout in Figure 5.3.10(c). Then, we can decide in polynomial time whether $\mathcal{L}_{np}$ is really correct.

By (A_2), Fact 7 and the last three lemmas, we can assume that there are two distinct countries c_7 and c_8 in $V(G) - U$ such that $\{c_1, c_2, c_5, c_7\}$ and $\{c_2, c_4, c_6, c_8\}$ are MC_4's in G, $c_7 \notin \mathcal{C}^G_{U,F}(\{c_1, c_2, c_5\})$, $c_8 \notin \mathcal{C}^G_{U,F}(\{c_2, c_4, c_6\})$, $\{c_7, c_4\} \notin E(G)$, and $\{c_8, c_1\} \notin E(G)$. Let $U_8 = U \cup \{c_7, c_8\}$, and F_8 be the set of the edges $\{d_1, d_2\}$ in $G - U_8$ such that $\{d_1, d_2, c_i, c_j\}$ is an MC_4 in G for some $c_i \in U_8$ and $c_j \in U_8$.

Let $\mathcal{L}_{p,d}$ be the layout of $G[U]$ in Figure 5.3.10(d). By Fact 7, every correct layout of $G[U]$ in which $G[C]$ is layouted as a pizza can be transformed to $\mathcal{L}_{p,d}$. We want to decide which one of $\mathcal{L}_{p,d}$ and $\mathcal{L}_{np}$ is correct. A necessary condition for $\mathcal{L}_{p,d}$ to be correct is that in the graph $G - (\mathcal{C}^G_U(\{c_1, c_5\}) \cup \mathcal{C}^G_U(\{c_4, c_6\})$, the neighbors of c_1 are only c_2, …, c_5, c_7, and the neighbors of c_4 are only c_1, c_2, c_3, c_6, c_8. Also note that the countries in $\mathcal{C}^G_U(\{c_1, c_5\}) \cup \mathcal{C}^G_U(\{c_4, c_6\}$ have no effect on distinguishing $\mathcal{L}_{p,d}$ and $\mathcal{L}_{np}$. Thus, we may assume that the following (A_4) holds:

(A_4) $N_G(c_1) = \{c_2, \ldots, c_5, c_7\}$ and $N_G(c_4) = \{c_1, c_2, c_3, c_6, c_8\}$.

To distinguish $\mathcal{L}_{p,d}$ and $\mathcal{L}_{np}$, we distinguish three cases as follows.

Case 1: $\{c_5, c_8\} \in E(G)$ or $\{c_6, c_7\} \in E(G)$. A simple inspection shows that if both $\{c_5, c_8\} \in E(G)$ and $\{c_6, c_7\} \in E(G)$, then $\mathcal{L}_{np}$ is incorrect. So, by symmetry, we may assume that $\{c_5, c_8\} \in E(G)$ but $\{c_6, c_7\} \notin E(G)$. Then, by (A_3) and (A_4), every correct layout of $G[U_8]$ in which C is layouted as a nonpizza can be transformed to the one in Figure 5.3.21-(b). Figure 5.3.21-(b) is explicit. We claim that the layout in Figure 5.3.21-(b) is correct iff $\{\mathcal{C}^G_{U_8,F_8}(W) \mid W \in \mathcal{W}\}$ is a partition of $V(G) - U_8$, where $\mathcal{W}$ is the class of the following sets: $\{c_2, c_5\}$, $\{c_2, c_6\}$, $\{c_2, c_8\}$, $\{c_5, c_7\}$, $\{c_6, c_8\}$, $\{c_7, c_8\}$, and $\{c_2, c_6, c_8\}$. The necessary condition is clear. To see the sufficient condition, assume that $\{\mathcal{C}^G_{U_8,F_8}(W) \mid W \in \mathcal{W}\}$ is a partition of $V(G) - U_8$. Then, every correct layout of $G[U_8]$ in which C is layouted as a pizza can be transformed to the layout in Figure 5.3.21-(a), and the latter can be transformed to the one in Figure 5.3.21-(b).

Case 2: $\{c_7, c_8\} \in E(G)$, $\{c_5, c_8\} \notin E(G)$, and $\{c_6, c_7\} \notin E(G)$. Then, by (A_3) and (A_4), every correct layoute of $G[U_8]$ in which $G[C]$ is layouted as a pizza can be transformed to one of the first two layouts in Figure 5.3.21. Both Figure 5.3.21(a) and Figure 5.3.21(b) are explicit.

Case 2.1: Either there is no $d \notin U_8$ such that $\{c_2, c_7, c_8, d\}$ is an MC_4 in G or the unique $d \notin U_8$ such that $\{c_2, c_7, c_8, d\}$ is an MC_4 in G belongs to $\mathcal{C}^G_{U_8,F_8}(\{c_2, c_7, c_8\})$. Then, every correct layout of $G[U_8]$ in which $G[C]$ is layouted as a pizza can be transformed to the one in Figure 5.3.21(a). We may assume that there is no $d \notin U_8$ such that $\{c_2, c_5, c_7, d\}$ or $\{c_2, c_6, c_8, d\}$ is an MC_4 in G; otherwise, the layout in Figure 5.3.21(a) is incorrect. Then, every correct layout of $G[U_8]$ in which $G[C]$ is layouted as a nonpizza can be transformed to the one in Figure 5.3.21(c). Figure 5.3.21(c) is explicit. It is easy to see that the layout in Figure 5.3.21(a) can be transformed to the one in Figure 5.3.21(c) iff $\mathcal{C}^G_{U_8,F_8}(W) = \emptyset$ for every $W \subseteq U_8$ such that $\{c_5, c_6\} \subseteq W$, $\{c_5, c_8\} \subseteq W$, or $\{c_6, c_7\} \subseteq W$.

Case 2.2: The unique $c_9 \notin U_8$ such that $\{c_2, c_7, c_8, c_9\}$ is an MC_4 in G does not belong to $\mathcal{C}^G_{U_8,F_8}(\{c_2, c_7, c_8\})$. Then, every correct layout of $G[U_8]$ in which $G[C]$ is layouted as a pizza can be transformed to the one in Figure 5.3.21(b). We may assume that $\mathcal{C}^G_{U_8}(\{c_2, c_6\}) = \emptyset$; otherwise, the layout in Figure 5.3.21(b) is incorrect and $\mathcal{L}_{np}$ must be correct. Let $U_9 = U_8 \cup \{c_9\}$, and F_9 be the set of the edges $\{d_1, d_2\}$ in $G - U_9$ such that $\{d_1, d_2, c_i, c_j\}$ is an MC_4 in G for some $c_i \in U_9$ and $c_j \in U_9$.

Case 2.2.1: Both $\{c_5, c_9\} \in E(G)$ and $\{c_6, c_9\} \in E(G)$. By (A_4), every correct layout of $G[U_9]$ in which $G[C]$ is layouted as a nonpizza can be transformed to the one in Figure 5.3.22(a). Figure 5.3.22(a) is explicit. We claim that the layout in Figure 5.3.22(a) is correct iff the family $\mathcal{F} = \{\mathcal{C}^G_{U_9}(W) \mid W \in \mathcal{W}\}$ is a partition of $V(G) - U_9$, where $\mathcal{W}$ is the class of the following sets: $\{c_2, c_5\}$, $\{c_2, c_9\}$, $\{c_5, c_7\}$, $\{c_6, c_8\}$, $\{c_7, c_8\}$, $\{c_7, c_9\}$, $\{c_8, c_9\}$, and $\{c_7, c_8, c_9\}$. The ncessary condition of this claim is clear from Figure 5.3.22(a). On the other hand, if $\mathcal{F}$ is a partition of $V(G) - U_9$, then every correct layout of $G[U_9]$ in which $G[C]$ is layouted as a pizza can be transformed to the one in Figure 5.3.22(b) which is explicit, and the latter can be transformed to the one in Figure 5.3.22(a).

Case 2.2.2: $\{c_5, c_9\} \notin E(G)$ but $\{c_6, c_9\} \in E(G)$. Then, we may assume that there is no $d \notin U_8$ such that $\{c_2, c_5, c_7, d\}$ or $\{c_2, c_7, c_9, d\}$ is an MC_4 in G; otherwise, the layout in Figure 5.3.21(b) is incorrect. By (A_4), every correct layout of $G[U_9]$ in which $G[C]$ is layouted as a nonpizza can be transformed to the one in Figure 5.3.23(a). Figure 5.3.23(a) is explicit except that the three countries c_7, c_8, and c_9 may meet each other at the same point.

No matter whether they meet or not, a necessary condition for the layout in Figure 5.3.23(a) to be correct is that in the graph $G - \mathcal{C}^G_{U_8}(\{c_6, c_8\})$, the neighbors of c_6 are only c_2, c_3, c_4, c_8, and c_9. Thus, every correct layout of $G[U_9]$ in which $G[C]$ is layouted as a pizza can be transformed to the one in Figure 5.3.23(b). We may assume that there is no $d \notin U_9$ such that $\{c_7, c_8, c_9, d\}$ is an MC_4 in G; otherwise, the layout in Figure 5.3.23(b) is incorrect. By this assumption, we can now assume that Figure 5.3.23(a) is actually explicit. We claim that the layout of $G[U_9]$ in Figure 5.3.23(a) is correct iff $\mathcal{C}^G_{U_9,F_9}(W) = \emptyset$ for every $W \subseteq U_9$ such that $\{c_5, c_6\} \subseteq W$, $\{c_5, c_9\} \subseteq W$, $\{c_6, c_7\} \subseteq W$, or $\{c_6, c_9\} \subseteq W$. The necessary condition of the claim is obvious. On the other hand, if $\mathcal{C}^G_{U_9,F_9}(W) = \emptyset$ for every such $W \subseteq U_9$, then every correct layout of $G[U_9]$ in which $G[C]$ is layouted as a pizza can be transformed to the one in Figure 5.3.23(b) which is explicit, and the latter can be transformed to the one in Figure 5.3.23(a).

Case 2.2.3: $\{c_5, c_9\} \in E(G)$ but $\{c_6, c_9\} \notin E(G)$. Similar to Case 2.2.2.

Case 2.2.4: $\{c_5, c_9\} \notin E(G)$ and $\{c_6, c_9\} \notin E(G)$. Then, every correct layout of $G[U_9]$ in which $G[C]$ is layouted as a pizza can be transformed to the one in Figure 5.3.24(a). Figure 5.3.24(a) is explicit. We may assume that there is no $d \notin U_8$ such that $\{c_2, c_5, c_7, d\}$ or $\{c_2, c_6, c_8, d\}$ is an MC_4 in G; otherwise, the layout in Figure 5.3.24(a) is incorrect. Also recall that $c_9 \notin \mathcal{C}^G_{U_8,F_8}(\{c_2, c_7, c_8\})$. Thus, by (A_4), every correct layout of $G[U_9]$ in which $G[C]$ is layouted as a nonpizza can be transformed to one of the last two layouts in Figure 5.3.24. Both Figure 5.3.24(b) and Figure 5.3.24(c) are explicit. Since $c_9 \notin \mathcal{C}^G_{U_8,F_8}(\{c_2, c_7, c_8\})$, $c_9 \in \mathcal{C}^G_{U_8,F_8}(\{c_2, c_7, c_8, c_6\})$ if the layout in Figure 5.3.24(b) is correct, and $c_9 \in \mathcal{C}^G_{U_8,F_8}(\{c_2, c_7, c_8, c_5\})$ if the layout in Figure 5.3.24(c) is correct. Thus, we may assume that either $c_9 \in \mathcal{C}^G_{U_8,F_8}(\{c_2, c_7, c_8, c_6\})$ or $c_9 \in \mathcal{C}^G_{U_8,F_8}(\{c_2, c_7, c_8, c_5\})$; otherwise, $\mathcal{L}_{np}$ is incorrect. Suppose that $c_9 \in \mathcal{C}^G_{U_8,F_8}(\{c_2, c_7, c_8, c_6\})$; the other case is similar. Then, every correct layout of $G[U_9]$ in which $G[C]$ is layouted as a nonpizza can be transformed to the one in Figure 5.3.24(b). Let K_9 be the connected component in $G - U_8 - F_8$ containing c_9. We claim that the layout of $G[U_9]$ in Figure 5.3.24(b) is correct iff (i) $K_9 \cap N_G(c_7) = \{c_9\}$ and (ii) the family $\mathcal{F} = \{\mathcal{C}^G_{U_8,F_8}(W) \mid W \in \mathcal{W}\}$ is a partition of $V(G) - (U_8 \cup K_9)$, where $\mathcal{W}$ is the class of the following sets: $\{c_2, c_5\}$, $\{c_2, c_7\}$, $\{c_2, c_6\}$, $\{c_5, c_7\}$, $\{c_6, c_8\}$, $\{c_7, c_8\}$, $\{c_2, c_5, c_7\}$. The necessary condition of the claim is obvious. To see the sufficient condition, it suffices to see that when (i) and (ii) hold, the layout in Figure 5.3.24(a) can be transformed to the layout in Figure 5.3.24(b).

Case 3: $\{c_7, c_8\} \notin E(G)$, $\{c_5, c_8\} \notin E(G)$, and $\{c_6, c_7\} \notin E(G)$. Then, by (A_3) and (A_4), every correct layout of $G[U_8]$ in which $G[C]$ is layouted as a pizza can be transformed to the one in Figure 5.3.25. Figure 5.3.25 is explicit. Recall that $c_7 \notin \mathcal{C}^G_{U,F}(\{c_1, c_2, c_5\})$, and $c_8 \notin \mathcal{C}^G_{U,F}(\{c_2, c_4, c_6\})$. Thus, the following (A_5) is a necessary condition for the layout in Figure 5.3.25 to be correct:

(A_5) $\mathcal{C}^G_{U_8}(\{c_2, c_5\}) = \mathcal{C}^G_{U_8}(\{c_2, c_6\}) = \emptyset$.

We may assume that (A_5) holds; otherwise, $\mathcal{L}_{np}$ must be correct. We define two Boolean variables b_5 and b_6 as follows. Let $b_5 = 1$ iff either there is no $d \notin U_8$ such that $\{d, c_2, c_5, c_7\}$ is an MC_4 in G or the unique $d \notin U$ such that $\{d, c_2, c_5, c_7\}$ is an MC_4 in G belongs to $\mathcal{C}^G_{U_8,F_8}(\{c_2, c_5, c_7\})$. Similarly, let $b_6 = 1$ iff either there is no $d \notin U_8$ such that $\{d, c_2, c_6, c_8\}$ is an MC_4 in G or the unique $d \notin U$ such that $\{d, c_2, c_6, c_8\}$ is an MC_4 in G belongs to $\mathcal{C}^G_{U_8,F_8}(\{c_2, c_6, c_8\})$.

Case 3.1: $b_5 = b_6 = 1$. Then, every correct layout of $G[U_8]$ in which $G[C]$ is layouted as a nonpizza can be transformed to the one in Figure 5.3.26. Figure 5.3.26 is explicit. Let $X = \mathcal{C}^G_{U_8,F_8}(\{c_2, c_5, c_7\})$, $Y = \mathcal{C}^G_{U,F}(\{c_2, c_7, c_8\})$, and $Z = \mathcal{C}^G_{U,F}(\{c_2, c_6, c_8\})$. We claim that the layout in Figure 5.3.26 is correct iff the following (i), (ii), and (iii) hold:

(i) The family $\mathcal{F} = \{\mathcal{C}^G_{U_8,F_8}(W) \mid W \in \mathcal{W}\}$ is a partition of $V(G) - U_8$, where $\mathcal{W}$ is the class of the following sets: $\{c_2, c_5\}$, $\{c_2, c_6\}$, $\{c_2, c_7\}$, $\{c_2, c_8\}$, $\{c_5, c_7\}$, $\{c_6, c_8\}$, $\{c_7, c_8\}$, $\{c_2, c_5, c_7\}$, $\{c_2, c_6, c_8\}$, and $\{c_2, c_7, c_8\}$.

(ii) If $N_G(X) \cap Y \neq \emptyset$, then $\mathcal{C}^G_{U_8,F_8}(\{c_2, c_7\}) = \emptyset$.

(iii) If $N_G(Z) \cap Y \neq \emptyset$, then $\mathcal{C}^G_{U_8,F_8}(\{c_2, c_8\}) = \emptyset$.

Case 3.2: $b_5 = 1$ but $b_6 = 0$. Then, by (A_5), every correct layout of $G[U_8]$ in which $G[C]$ is layouted as a nonpizza can be transformed to the one in Figure 5.3.27. Figure 5.3.27 is explicit. Moreover, there is a country $c_9 \notin U_8$ such that $\{c_2, c_6, c_8, c_9\}$ is an MC_4 in G and $c_9 \notin \mathcal{C}^G_{U_8,F_8}(\{c_2, c_6, c_8\})$. If the layout in Figure 5.3.27 is correct, then in the graph $G - \mathcal{C}^G_{U_8}(\{c_6, c_8\}))$, the neighbors of c_6 are only c_2, c_3, c_4, c_8, and c_9. Also note that the countries in $\mathcal{C}^G_{U_8}(\{c_6, c_8\})$ have no effect on distinguishing the layout in Figure 5.3.25 and the layout in Figure 5.3.27. So, we may assume that the following (B_1) holds:

(B_1) $N_G(c_6) = \{c_2, c_3, c_4, c_8, c_9\}$.

Let $U_9 = U_8 \cup \{c_9\}$, and F_9 be the set of the edges $\{d_1, d_2\}$ in $G - U_9$ such that $\{d_1, d_2, c_i, c_j\}$ is an MC_4 in G for some $\{c_i, c_j\} \subseteq U_9$.

Case 3.2.1: $\{c_7, c_9\} \in E(G)$, and either there is no $d \notin U_9$ such that $\{c_2, c_7, c_9, d\}$ is an MC_4 in G or the unique $d \notin U_9$ such that $\{c_2, c_7, c_9, d\}$ is an MC_4 in G belongs to $\mathcal{C}^G_{U_9,F_9}(\{c_2, c_7, c_9\})$. Then, every correct layout of $G[U_9]$ in which $G[C]$ is layouted as a nonpizza can be transformed to the one in Figure 5.3.28(b). Figure 5.3.28(b) is explicit. We claim that the layout in Figure 5.3.28(b) is correct iff the family $\mathcal{F} = \{\mathcal{C}^G_{U_9,F_9}(W) \mid W \in \mathcal{W}\}$ is a partition of $V(G) - U_9$, where $\mathcal{W}$ is the class of the following sets: $\{c_2, c_5\}$, $\{c_2, c_7\}$, $\{c_2, c_9\}$, $\{c_5, c_7\}$, $\{c_7, c_8\}$, $\{c_7, c_9\}$, $\{c_8, c_9\}$, $\{c_2, c_5, c_7\}$, $\{c_2, c_7, c_9\}$, and $\{c_7, c_8, c_9\}$. The necessary condition of the claim is obvious. On the other hand, if $\mathcal{F}$ is a partition of $V(G) - U_9$, then every correct layout of $G[U_9]$ in which $G[C]$ is layouted as a pizza can be transformed to the one in Figure 5.3.28(a) which is explicit, and the latter can be transformed to the one in Figure 5.3.28(b).

Case 3.2.2: $\{c_7, c_9\} \in E(G)$, and the unique $c_{10} \notin U_9$ such that $\{c_2, c_7, c_9, c_{10}\}$ is an MC_4 in G does not belong to $\mathcal{C}^G_{U_9,F_9}(\{c_2, c_7, c_9\})$. Then, by (A_3), (A_4), and (B_1), $c_{10} \in \mathcal{C}^G_{U_9,F_9}(\{c_2, c_7, c_9\} \cup W)$ for some nonempty $W \subseteq \{c_5, c_8\}$. Let $U_{10} = U_9 \cup \{c_{10}\}$. Let K_{10} be the connected component in $G - U_9$ containing c_{10}.

Case 3.2.2.1: $c_{10} \in \mathcal{C}^G_{U_9,F_9}(\{c_2, c_7, c_9, c_5\})$. Then, every correct layout of $G[U_{10}]$ in which $G[C]$ is layouted as a nonpizza can be transformed to the one in Figure 5.3.29(b). Figure 5.3.29(b) is explicit except that c_{10} may touch c_5. We claim that the layout in Figure 5.3.29(b) is correct iff $K_{10} \cap N_G(c_9) = \{c_{10}\}$ and the family $\mathcal{F} = \{\mathcal{C}^G_{U_9}(W) \mid W \in \mathcal{W}\}$ is a partition of $V(G) - (U_9 \cup K_{10})$, where $\mathcal{W}$ is the class of the following sets: $\{c_2, c_9\}$, $\{c_5, c_7\}$, $\{c_7, c_8\}$, $\{c_7, c_9\}$, $\{c_8, c_9\}$, and $\{c_7, c_8, c_9\}$. The necessary condition of the claim is obvious. On the other hand, if $K_{10} \cap N_G(c_9) = \{c_{10}\}$ and $\mathcal{F}$ is a partition of $V(G) - (U_9 \cup K_{10})$, then every correct layout of $G[U_9 \cup K_{10}]$ in which $G[C]$ is layouted as a pizza can be transformed to the one in Figure 5.3.29(a), and the latter can be transformed to the layout in Figure 5.3.29(b).

Case 3.2.2.2: $c_{10} \in \mathcal{C}^G_{U_9,F_9}(\{c_2, c_7, c_9, c_8\})$ but $\{c_8, c_{10}\} \notin E(G)$. Then, every correct layout of $G[U_{10}]$ in which $G[C]$ is layouted as a nonpizza can be transformed to the one in Figure 5.3.30(b). Figure 5.3.30(b) is explicit. We may assume that there is no $d \notin U_{10}$ such that $\{c_2, c_8, c_9, d\}$ is an MC_4 in G; otherwise, the layout in Figure 5.3.30(b) is incorrect. Then, we claim that the layout in Figure 5.3.30(b) is correct iff $K_{10} \cap N_G(c_2) = \{c_{10}\}$ and the family $\mathcal{F} = \{\mathcal{C}^G_{U_9}(W) \mid W \in \mathcal{W}\}$ is a partition of $V(G) - (U_9 \cup K_{10})$, where $\mathcal{W}$ is the class of the following sets: $\{c_2, c_7\}$, $\{c_2, c_9\}$, $\{c_5, c_7\}$, $\{c_7, c_8\}$, $\{c_8, c_9\}$, and $\{c_2, c_5, c_7\}$. The necessary condition of the claim is obvious. On the other hand, if $K_{10} \cap N_G(c_9) = \{c_{10}\}$ and $\mathcal{F}$ is a partition of $V(G) - (U_9 \cup K_{10})$, then every correct layout of $G[U_9 \cup K_{10}]$ in which $G[C]$ is layouted as a pizza can be transformed to the one in Figure 5.3.30(a) which is explicit, and the latter can be transformed to the layout in Figure 5.3.30(b).

Case 3.2.2.3: $c_{10} \in \mathcal{C}^G_{U_9,F_9}(\{c_2, c_7, c_9, c_8\})$ and $\{c_8, c_{10}\} \in E(G)$. Then, every correct layout of $G[U_{10}]$ in which $G[C]$ is layouted as a nonpizza can be transformed to the one in Figure 5.3.31(c) or the one in Figure 5.3.31(d). Both Figure 5.3.31(c) and Figure 5.3.31(d) are explicit. Two common necessary conditions for the two layouts in Figure 5.3.31(c) and Figure 5.3.31(d) to be correct are the following (B_1') and (B_2'):

(B_1') In the graph $G - \{c_1, c_3, \ldots, c_9\} - \{\{c_2, c_{10}\}\}$, c_2 cannot reach c_{10}.

(B_2') If there is a country $d \notin U_{10}$ such that $\{d, c_7\} \in E(G)$ and $\{d, c_9\} \in E(G)$, then $\{d, c_8\} \in E(G)$.

We may assume that (B_1') and (B_2') hold; otherwise, $\mathcal{L}_{np}$ is incorrect. Then, every correct layout of $G[U_{10}]$ in which $G[C]$ is layouted as a pizza can be transformed to the one in Figure 5.3.31(a) or the one in Figure 5.3.31(b). Both Figure 5.3.31(a) and Figure 5.3.31(b) are explicit. A common necessary condition for the two layouts in Figure 5.3.31(a) and Figure 5.3.31(b) to be correct is the following (B_3'):

(B_3') There is no $d \notin U_{10}$ such that both $\{c_8, c_9, c_{10}, d\}$ is an MC_4 in G and $d \notin \mathcal{C}^G_{U_{10}}(\{c_8, c_9, c_{10}\})$.

We may assume that (B_3') holds; otherwise, $\mathcal{L}_{np}$ must be correct. Then, every correct layout of $G[U_{10}]$ in which $G[C]$ is layouted as a nonpizza can be transformed to the one in Figure 5.3.31(c). Let F_{10} be the set of the edges $\{d_1, d_2\}$ in $G - U_{10}$ such that $\{d_1, d_2, c_i, c_j\}$ is an MC_4 in G for some $c_i \in U_{10}$ and $c_j \in U_{10}$. It is easy to see that the layout in Figure 5.3.31(c) is correct iff the family $\{\mathcal{C}^G_{U_{10},F_{10}}(W) \mid W \in \mathcal{W}\}$ is a partition of $V(G) - U_{10}$, where $\mathcal{W}$ is the class of the following sets: $\{c_2, c_5\}$, $\{c_2, c_7\}$, $\{c_2, c_9\}$, $\{c_5, c_7\}$, $\{c_7, c_8\}$, $\{c_7, c_{10}\}$, $\{c_8, c_9\}$, $\{c_8, c_{10}\}$, $\{c_2, c_5, c_7\}$, $\{c_7, c_8, c_{10}\}$, and $\{c_8, c_9, c_{10}\}$.

Case 3.2.2.4: $c_{10} \in \mathcal{C}^G_{U_9,F_9}(\{c_2, c_7, c_9, c_5, c_8\})$. Then, the layout in Figure 5.3.27 cannot be correct and so $\mathcal{L}_{np}$ is incorrect.

Case 3.2.3 $\{c_7, c_9\} \notin E(G)$. Then, every correct layout of $G[U_9]$ in which $G[C]$ is layouted as a nonpizza can be transformed to the one in Figure 5.3.32. Figure 5.3.32 is explicit. Since $c_9 \notin \mathcal{C}^G_{U_8,F_8}(\{c_2, c_6, c_8\})$, a necessary condition for the layout in Figure 5.3.32 to be correct is the following (B_4'):

(B_4') $\mathcal{C}^G_{U_9}(\{c_2, c_8\}) = \emptyset$.

We may assume that (B_4') holds; otherwise, $\mathcal{L}_{np}$ is incorrect.

Case 3.2.3.1: Either there is no $d \notin U_9$ such that $\{c_2, c_8, c_9, d\}$ is an MC_4 in G or the unique $d \notin U_9$ such that $\{c_2, c_8, c_9, d\}$ is an MC_4 in G belongs to $\mathcal{C}^G_{U_9,F_9}(\{c_2, c_8, c_9\})$. Then, every correct layout of $G[U_9]$ in which $G[C]$ is layouted as a pizza can be transformed to the one in Figure 5.3.33. Figure 5.3.33 is explicit. Let $X = \mathcal{C}^G_{U_9,F_9}(\{c_2, c_5, c_7\})$, $Y =$

$\mathcal{C}^G_{U_9,F_9}(\{c_2, c_7, c_9\})$, and $Z = \mathcal{C}^G_{U_9,F_9}(\{c_2, c_8, c_9\})$. Then, it is easy to see that the layout in Figure 5.3.33 can be transformed to the layout in Figure 5.3.32 iff the following (1) through (4) hold.

(1) $\mathcal{C}^G_{U_9,F_9}(W) = \emptyset$ for every $W \subseteq U$ such that $\{c_5, c_9\} \subseteq W$.

(2) If $Y \neq \emptyset$, then $\mathcal{C}^G_{U_9,F_9}(\{c_2, c_8\}) = \emptyset$.

(3) If $N_G(X) \cap Y \neq \emptyset$, then $\mathcal{C}^G_{U_9,F_9}(\{c_2, c_7\}) = \emptyset$.

(4) If $Y \neq \emptyset$ and $Z \neq \emptyset$, $|N_G(Z) \cap Y| = |Z \cap N_G(Y)| = 1$, $\{c_2, c_9\} \cup (N_G(Z) \cap Y) \cup (Z \cap N_G(Y))$ is an MC$_4$ in G, $Z \cap N_G(c_2) = Z \cap N_G(Y)$, and $N_G(c_9) \cap Y = N_G(Z) \cap Y$.

Case 3.2.3.2: The unique $c_{10} \notin U_9$ such that $\{c_2, c_8, c_9, c_{10}\}$ is an MC$_4$ in G does not belong to $\mathcal{C}^G_{U_9,F_9}(\{c_2, c_8, c_9\})$. Then, by (B_1) and (B'_4), every correct layout of $G[U_9]$ in which $G[C]$ is layouted as a pizza can be transformed to the one in Figure 5.3.34. Figure 5.3.34 is explicit. If the layout in Figure 5.3.34 is correct, then in the graph $G - \mathcal{C}^G_{U_9}(\{c_8, c_9\})$, the neighbors of c_8 are only c_2, c_4, c_6, c_9, and c_{10}. Also note that the countries $\mathcal{C}^G_{U_9}(\{c_8, c_9\})$ have no effect on distinguishing the two layouts in Figuer 5.3.32 and Figuer 5.3.34. Thus, we may assume that the following (B'_5) holds:

(B'_5) $N_G(c_8) = \{c_2, c_4, c_6, c_9, c_{10}\}$.

Now, we construct a new graph G' from G by deleting c_6 and c_8 and adding the three new edges $\{c_9, c_3\}$, $\{c_9, c_4\}$, and $\{c_{10}, c_4\}$. Let $U' = \{c_1, \ldots, c_5, c_9\}$, and F' be the set of the edges $\{d_1, d_2\}$ in $G' - U'$ such that $\{d_1, d_2, c_i, c_j\}$ is an MC$_4$ in G' for some $c_i \in U'$ and $c_j \in U'$. Then, $c_7 \notin \mathcal{C}^{G'}_{U',F'}(\{c_1, c_2, c_5\})$ and $c_{10} \notin \mathcal{C}^{G'}_{U',F'}(\{c_2, c_4, c_9\})$. Figure 5.3.35 shows two layouts of $G'[U']$. Both figures in Figure 5.3.35 are explicit. By (B_1), (B'_5), and the construction of G', the following (i) and (ii) hold:

(i) The layout of $G[U_9]$ in Figure 5.3.34 is correct iff the layout of $G'[U']$ in Figure 5.3.35(a) is correct.

(ii) The layout of $G[U_9]$ in Figure 5.3.32 is correct iff the layout of $G'[U']$ in Figure 5.3.35(b) is correct.

Thus, it suffices to distinguish the two layouts in Figure 5.3.35. Note that distinguishing the two layouts in Figure 5.3.35 is a sub-task of distingushing $\mathcal{L}_{p,d}$ and $\mathcal{L}_{np}$, because $\{c_1, c_2, c_5, c_7\}$ and $\{c_2, c_4, c_9, c_{10}\}$ are MC$_4$'s in G', $c_7 \notin \mathcal{C}^{G'}_{U',F'}(\{c_1, c_2, c_5\})$, and $c_{10} \notin \mathcal{C}^{G'}_{U',F'}(\{c_2, c_4, c_9\})$. Hence, we may recursively distinguish the two layouts in Figure 5.3.35.

Case 3.3: $b_5 = 0$ but $b_6 = 1$. Similar to Case 3.2.

Case 3.4: $b_5 = b_6 = 0$. Then, by (A_5), every correct layout of $G[U_8]$ in which $G[C]$ is layouted as a nonpizza can be transformed to the one in Figure 5.3.36. Figure 5.3.36 is explicit. Moreover, there is a country $c_9 \notin U_8$ such that $\{c_2, c_5, c_7, c_9\}$ is an MC$_4$ in G and $c_9 \notin \mathcal{C}^G_{U_8,F_8}(\{c_2, c_5, c_7\})$, and there is a country $c_{10} \notin U_8$ such that $\{c_2, c_6, c_8, c_{10}\}$ is an MC$_4$ in G and $c_{10} \notin \mathcal{C}^G_{U_8,F_8}(\{c_2, c_6, c_8\})$. If the layout in Figure 5.3.36 is correct, then in the graph $G - (\mathcal{C}^G_{U_8}(\{c_5, c_7\}) \cup \mathcal{C}^G_{U_8}(\{c_6, c_8\}))$, the neighbors of c_5 are only c_1, c_2, c_3, c_7, c_9, and the neighbors of c_6 are only c_2, c_3, c_4, c_8, and c_{10}. Also note that the countries in $\mathcal{C}^G_{U_8}(\{c_5, c_7\}) \cup \mathcal{C}^G_{U_8}(\{c_6, c_8\})$ have no effect on distinguishing the layout in Figure 5.3.25 and the layout in Figure 5.3.36. So, we may assume that the following (B_2) holds:

(B_2) $N_G(c_5) = \{c_1, c_2, c_3, c_7, c_9\}$ and $N_G(c_6) = \{c_2, c_3, c_4, c_8, c_{10}\}$.

Case 3.4.1: $c_9 = c_{10}$. Let $U_9 = U_8 \cup \{c_9\}$. Then, every correct layout of $G[U_9]$ in which $G[C]$ is layouted as a nonpizza can be transformed to the one in Figure 5.3.37(b). Figure 5.3.37(b) is explicit. We claim that the layout in Figure 5.3.37(b) is correct iff the family $\mathcal{F} = \{\mathcal{C}^G_{U_9}(W) \mid W \in \mathcal{W}\}$ is a partition of $V(G) - U_9$, where $\mathcal{W}$ is the class of the following sets: $\{c_2, c_9\}$, $\{c_7, c_8\}$, $\{c_7, c_9\}$, $\{c_8, c_9\}$, and $\{c_7, c_8, c_9\}$. The necessary condition of the claim is obvious. On the other hand, if $\mathcal{F}$ is a parition of $V(G) - U_9$, then every correct layout of $G[U_9]$ in which $G[C]$ is layouted as a pizza can be transformed to the one in Figure 5.3.37(a) which is explicit, and the latter can be transformed to the one in Figure 5.3.37(b).

Case 3.4.2: $c_9 \neq c_{10}$. Let $U_{10} = U_8 \cup \{c_9, c_{10}\}$, and F_{10} be the set of the edges $\{d_1, d_2\}$ in $G - U_{10}$ such that $\{d_1, d_2, c_i, c_j\}$ is an MC_4 in G for some $c_i \in U_{10}$ and $c_j \in U_{10}$.

Case 3.4.2.1: $\{c_9, c_8\} \in E(G)$. Then, every correct layout of $G[U_{10}]$ in which $G[C]$ is layouted as a pizza can be transformed to the one in Figure 5.3.38(a). Figure 5.3.38(a) is explicit. We may assume that $\{c_9, c_{10}\} \in E(G)$, $\{c_7, c_{10}\} \notin E(G)$, and there is no $d \notin U_{10}$ such that $\{c_2, c_9, c_{10}, d\}$ or $\{c_8, c_9, c_{10}, d\}$ is an MC_4 in G; otherwise, the layout in Figure 5.3.38(a) is incorrect and so $\mathcal{L}_{np}$ must be correct. Then, every correct layout of $G[U_{10}]$ in which $G[C]$ is layouted as a nonpizza can be transformed to the one in Figure 5.3.38(b). Figure 5.3.38(b) is explicit. Then, it is easy to see that the layout in Figure 5.3.38(a) can be transformed to the one in Figure 5.3.38(b) iff $\mathcal{C}^G_{U_{10},F_{10}}(\{c_2, c_7\})$, $\mathcal{C}^G_{U_{10},F_{10}}(\{c_2, c_8\})$, and $\mathcal{C}^G_{U_{10},F_{10}}(\{c_2, c_7, c_9\})$ are all empty.

Case 3.4.2.2: $\{c_{10}, c_7\} \in E(G)$. Similar to Case 3.4.2.1.

Case 3.4.2.3: $\{c_9, c_8\} \notin E(G)$ and $\{c_{10}, c_7\} \notin E(G)$ but $\{c_9, c_{10}\} \in E(G)$. Then, every correct layout of $G[U_{10}]$ in which $G[C]$ is layouted as a nonpizza can be transformed to the one in Figure 5.3.39(a) or the one in Figure 5.3.39(b). Both Figure 5.3.39(a) and Figure 5.3.39(b) are explicit.

Case 3.4.2.3.1: Either there is no $d \notin U_{10}$ such that $\{c_2, c_9, c_{10}, d\}$ is an MC_4 in G or the unique $d \notin U_{10}$ such that $\{c_2, c_9, c_{10}, d\}$ is an MC_4 in G belongs to $\mathcal{C}^G_{U_{10},F_{10}}(\{c_2, c_9, c_{10}\})$. Then, every correct layout of $G[U_{10}]$ in which $G[C]$ is layouted as a nonpizza can be transformed to the one in Figure 5.3.39(a). We claim that the layout in Figure 5.3.39(a) is correct iff $\mathcal{C}^G_{U_{10},F_{10}}(W) = \emptyset$ for every $W \subseteq U_{10}$ such that $\{c_2, c_7\} \subseteq W$ or $\{c_2, c_8\} \subseteq W$. The necessary condition of the claim is obvioius. On the other hand, if $\mathcal{C}^G_{U_{10},F_{10}}(W) = \emptyset$ for every such W, then every correct layout of $G[U_{10}]$ in which $G[C]$ is layouted as a pizza can be transformed to the one in Figure 5.3.45 which is explicit, and the latter can be transformed to the one in Figure 5.3.39(a).

Case 3.4.2.3.2: The unique $c_{11} \notin U_{10}$ such that $\{c_2, c_9, c_{10}, c_{11}\}$ is an MC_4 in G does not belong to $\mathcal{C}^G_{U_{10},F_{10}}(\{c_2, c_9, c_{10}\})$ and is adjacent to neither c_7 nor c_8 in G. Let $U_{11} = U_{10} \cup \{c_{11}\}$. Then, every correct layout of $G[U_{11}]$ in which $G[C]$ is layouted as a nonpizza can be transformed to the one in Figure 5.3.41(a). We may assume that there is no $d \notin U_{11}$ such that $\{c_2, c_7, c_9, d\}$ or $\{c_2, c_8, c_{10}, d\}$ is an MC_4 in G; otherwise, the layout in Figure 5.3.41(a) is incorrect. Then, by (A_3) and (B_2), every correct layout of $G[U_{11}]$ in which $G[C]$ is layouted as a pizza can be transformed to the one in Figure 5.3.41(b) or the one in Figure 5.3.41(c). Let K_{11} be the connected component in $G - U_{10} - F_{10}$ containing c_{11}. Then, it is not difficult to see that the two layouts in Figure 5.3.41(b) and Figure 5.3.41(c) can be transformed to the layout in Figure 5.3.41(a) iff $K_{11} \cap N_G(c_2) = \{c_{11}\}$ and $\mathcal{C}^G_{U_{10},F_{10}}(W) = \emptyset$ for every set W among $\{c_2, c_7\}$, $\{c_2, c_8\}$, $\{c_2, c_7, c_9\}$, and $\{c_2, c_8, c_{10}\}$.

Case 3.4.2.3.3: The unique $c_{11} \notin U_{10}$ such that $\{c_2, c_9, c_{10}, c_{11}\}$ is an MC_4 in G is adjacent to both c_7 and c_8 in G. Let $U_{11} = U_{10} \cup \{c_{11}\}$, F_{11} be the set of the edges $\{d_1, d_2\}$ in $G - U_{11}$ such that $\{d_1, d_2, c_i, c_j\}$ is an MC_4 in G for some $\{c_i, c_j\} \subseteq U_{11}$. Then, every correct layout of $G[U_{11}]$ in which $G[C]$ is layouted as a pizza can be transformed to the one in Figure 5.3.42(a). We may assume that there is no $d \notin U_{11}$ such that $\{c_7, c_9, c_{11}, d\}$ or $\{c_8, c_{10}, c_{11}, d\}$ is an MC_4 in G; otherwise, the layout in Figure 5.3.42(a) is incorrect and $\mathcal{L}_{np}$ must be correct. Then, every correct layout of $G[U_{11}]$ in which $G[C]$ is layouted as a nonpizza can be transformed to the one in Figure 5.3.42(b). Figure 5.3.42(b) is explicit. It is not difficult to see that the layout in Figure 5.3.42(a) can be transformed to the layout in Figure 5.3.42(b) iff $\mathcal{C}^G_{U_{11},F_{11}}(W) = \emptyset$ for every set W among $\{c_2, c_7\}$, $\{c_2, c_8\}$, $\{c_2, c_{11}\}$, $\{c_9, c_{10}\}$, and $\{c_9, c_{10}, c_{11}\}$.

Case 3.4.2.3.4: The unique $c_{11} \notin U_{10}$ such that $\{c_2, c_9, c_{10}, c_{11}\}$ is an MC_4 in G is adjacent to c_7 but not to c_8 in G. Let $U_{11} = U_{10} \cup \{c_{11}\}$, F_{11} be the set of the edges $\{d_1, d_2\}$ in $G - U_{11}$ such that $\{d_1, d_2, c_i, c_j\}$ is an MC_4 in G for some $\{c_i, c_j\} \subseteq U_{11}$. Then, every correct layout of $G[U_{11}]$ in which $G[C]$ is layouted as a nonpizza can be transformed to one of the two layouts in Figure 5.3.43. Both Figure 5.3.43(a) and Figure 5.3.43(b) are explicit.

Case 3.4.2.3.4.1: Either there is no $d \notin U_{11}$ such that $\{c_7, c_9, c_{11}, d\}$ is an MC_4 in G or the unique $d \notin U_{11}$ such that $\{c_7, c_9, c_{11}, d\}$ is an MC_4 in G belongs to $\mathcal{C}^G_{U_{11},F_{11}}(\{c_7, c_9, c_{11}\})$. Then, every correct layout of $G[U_{11}]$ in which $G[C]$ is layouted as a nonpizza can be transformed to the one in Figure 5.3.43(a). We claim that the layout in Figure 5.3.43(a) is correct iff $\mathcal{C}^G_{U_{11},F_{11}}(W) = \emptyset$ for every set W such that $\{c_2, c_7\} \subseteq W$, $\{c_2, c_8\} \subseteq W$, $\{c_2, c_{11}\} \subseteq W$, or $\{c_9, c_{10}\} \subseteq W$. The necessary condition is clear. To see the sufficient condition, assume that $\mathcal{C}^G_{U_{11},F_{11}}(W) = \emptyset$ for every such set W. Then, every correct layout of $G[U_{11}]$ in which $G[C]$ is layouted as a pizza can be transformed to one of the two layouts in Figure 5.3.44. Both Figure 5.3.44(a) and Figure 5.3.44(b) are explicit. Clearly, the two layouts in Figure 5.3.44 can be transformed to the one in Figure 5.3.43(a).

Case 3.4.2.3.4.2: The unique $c_{12} \notin U_{11}$ such that $\{c_7, c_9, c_{11}, c_{12}\}$ is an MC_4 in G does not belong to $\mathcal{C}^G_{U_{11},F_{11}}(\{c_7, c_9, c_{11}\})$. Then, every correct layout of $G[U_{11}]$ in which $G[C]$ is layouted as a nonpizza can be transformed to the one in Figure 5.3.43(b). By (A_4), we may assume that $c_{12} \in \mathcal{C}^G_{U_{11},F_{11}}(\{c_7, c_9, c_{11}\} \cup W)$ for some nonempty $W \subseteq \{c_8, c_{10}\}$; otherwise, the layout in Figure 5.3.43(b) is incorrect. Then, an easy inspection shows that $G[U_{11} \cup \{c_{12}\}]$ has no correct layout in which C is layouted as a pizza. Hence, $\mathcal{L}_{np}$ must be correct.

Case 3.4.2.3.5: The unique $c_{11} \notin U_{10}$ such that $\{c_2, c_9, c_{10}, c_{11}\}$ is an MC_4 in G is adjacent to c_8 but not to c_7 in G. Similar to Case 3.4.2.3.4.

Case 3.4.2.4: $\{c_9, c_8\} \notin E(G)$, $\{c_{10}, c_7\} \notin E(G)$, and $\{c_9, c_{10}\} \notin E(G)$. Then, every correct layout of $G[U_{10}]$ in which $G[C]$ is layouted as a nonpizza can be transformed to the one in Figure 5.3.45. Figure 5.3.45 is explicit. Since $c_9 \notin \mathcal{C}^G_{U_8,F_8}(\{c_2, c_5, c_7\})$ and $c_{10} \notin \mathcal{C}^G_{U_8,F_8}(\{c_2, c_6, c_8\})$, a necessary condition for the layout in Figure 5.3.45 to be correct is the following (B'_6):

(B'_6) $\mathcal{C}^G_{U_{10}}(\{c_2, c_7\}) = \mathcal{C}^G_{U_{10}}(\{c_2, c_8\}) = \emptyset$.

We may assume that (B'_6) holds; otherwise, $\mathcal{L}_{np}$ is incorrect. We define two Boolean variables b_7 and b_8 as follows. Let $b_7 = 1$ iff either there is no $d \notin U_{10}$ such that $\{c_2, c_7, c_9, d\}$ is an MC_4 in G or the unique $d \notin U_{10}$ such that $\{c_2, c_7, c_9, d\}$ is an MC_4 in G belongs to $\mathcal{C}^G_{U_{10},F_{10}}(\{c_2, c_7, c_9\})$. Similarly, let $b_8 = 1$ iff either there is no $d \notin U_{10}$ such that $\{c_2, c_8, c_{10}, d\}$ is an MC_4 in G or the unique $d \notin U_{10}$ such that $\{c_2, c_8, c_{10}, d\}$ is an MC_4 in G belongs to $\mathcal{C}^G_{U_{10},F_{10}}(\{c_2, c_8, c_{10}\})$.

Case 3.4.2.4.1: $b_7 = 1$ and $b_8 = 1$. Then, by (B_2), every correct layout of $G[U_{10}]$ in which $G[C]$ is layouted as a pizza can be transformed to the one in Figure 5.3.46. Figure 5.3.46 is explicit. Let $X = \mathcal{C}^G_{U_{10},F_{10}}(\{c_2, c_7, c_9\})$, $Y = \mathcal{C}^G_{U_{10},F_{10}}(\{c_2, c_9, c_{10}\})$, and $Z = \mathcal{C}^G_{U_{10},F_{10}}(\{c_2, c_8, c_{10}\})$. Then, by (B_2), it is easy to see that the layout in Figure 5.3.46 can be transformed to the layout in Figure 5.3.45 iff the following (1), (2), and (3) hold.

(1) If $Y \neq \emptyset$, then $\mathcal{C}^G_{U_{10},F_{10}}(\{c_2, c_7\}) = \emptyset$ and $\mathcal{C}^G_{U_{10},F_{10}}(\{c_2, c_8\}) = \emptyset$.

(2) If $X \neq \emptyset$ and $Y \neq \emptyset$, $|N_G(X) \cap Y| = |X \cap N_G(Y)| = 1$, $\{c_2, c_9\} \cup (N_G(X) \cap Y) \cup (X \cap N_G(Y))$ is an MC_4 in G, $N_G(c_2) \cap X = X \cap N_G(Y)$, and $N_G(c_9) \cap Y = Y \cap N_G(X)$.

(3) If $Z \neq \emptyset$ and $Y \neq \emptyset$, $|N_G(Z) \cap Y| = |Z \cap N_G(Y)| = 1$, $\{c_2, c_{10}\} \cup (N_G(Z) \cap Y) \cup (Z \cap N_G(Y))$ is an MC_4 in G, $N_G(c_2) \cap Z = Z \cap N_G(Y)$, and $N_G(c_{10}) \cap Y = Y \cap N_G(Z)$.

Case 3.4.2.4.2: $b_7 = 0$ or $b_8 = 0$. We assume that $b_8 = 0$; the case where $b_7 = 0$ is similar. Then, there is a country $c_{11} \notin U_{10}$ such that $\{c_2, c_8, c_{10}, c_{11}\}$ is an MC_4 in G does not belong to $\mathcal{C}^G_{U_{10},F_{10}}(\{c_2, c_8, c_{10}\})$. Then, by (B_2) and (B'_6), every correct layout of $G[U_8 \cup \{c_{10}\}]$ in which $G[C]$ is layouted as a pizza (or nonpizza, respectively) can be transformed to the one in Figure 5.3.47(a) (respectively, Figure 5.3.47(b)). Both figures in Figure 5.3.47 are explicit. If the layout in Figure 5.3.47(a) is correct, then in the graph $G - \mathcal{C}^G_{U_8 \cup \{c_{10}\}}(\{c_8, c_{10}\})$, the neighbors of c_8 are only c_2, c_4, c_6, c_{10}, and c_{11}. Also note that the countries $\mathcal{C}^G_{U_8 \cup \{c_{10}\}}(\{c_8, c_{10}\})$ have no effect on distinguishing the two layouts in Figuer 5.3.47. Thus, we may assume that the following (B'_7) holds:

(B'_7) $N_G(c_8) = \{c_2, c_4, c_6, c_{10}, c_{11}\}$.

Now, we construct a new graph G' from G by deleting c_6 and c_8 and adding the three new edges $\{c_{10}, c_3\}$, $\{c_{10}, c_4\}$, and $\{c_{11}, c_4\}$. Let $U' = \{c_1, \ldots, c_5, c_{10}\}$, and F' be the set of the edges $\{d_1, d_2\}$ in $G' - U'$ such that $\{d_1, d_2, c_i, c_j\}$ is an MC_4 in G' for some $c_i \in U'$ and $c_j \in U'$. Then, $c_7 \notin \mathcal{C}^{G'}_{U',F'}(\{c_1, c_2, c_5\})$ and $c_{11} \notin \mathcal{C}^{G'}_{U',F'}(\{c_2, c_4, c_{10}\})$. Figure 5.3.48 shows two layouts of $G'[U']$. Both figures in Figure 5.3.48 are explicit. By (B_2), (B'_7), and the construction of G', the following (i) and (ii) hold:

(i) The layout of $G[U_8 \cup \{c_{10}\}]$ in Figure 5.3.47(a) is correct iff the layout of $G'[U']$ in Figure 5.3.48(a) is correct.

(ii) The layout of $G[U_8 \cup \{c_{10}\}]$ in Figure 5.3.47(b) is correct iff the layout of $G'[U']$ in Figure 5.3.48(b) is correct.

Thus, it suffices to distinguish the two layouts in Figure 5.3.48. Note that distinguishing the two layouts in Figure 5.3.48 is a sub-task of distingushing $\mathcal{L}_{p,d}$ and $\mathcal{L}_{np}$, because $\{c_1, c_2, c_5, c_7\}$ and $\{c_2, c_4, c_{10}, c_{11}\}$ are MC_4's in G', $c_7 \notin \mathcal{C}^{G'}_{U',F'}(\{c_1, c_2, c_5\})$, and $c_{11} \notin \mathcal{C}^{G'}_{U',F'}(\{c_2, c_4, c_{10}\})$. Hence, we may recursively distinguish the two layouts in Figure 5.3.48.

5.4 Distinguishing pizzas and 1-type nonpizzas

Assume that the best nonpizza layout of C is a 1-type nonpizza. Our goal is to decide whether there is a correct 1-type nonpizza and to construct one if there is some. By the discussions in § 5.1, § 5.2, and § 5.3, we may assume that every MC_4 in G whose best nonpizza layout is not of 1-type has no correct nonpizza layout.

Since the best nonpizza layout of C is a 1-type nonpizza, there is exactly one MC_4 C_1 in G 3-sharing with C such that the unique element in $C - C_1$ and the unique element in $C_1 - C$ belong to the same connected component in the graph $G' = G - (C \cap C_1) -$

$\{\{d_1, d_2\} \mid \{d_1, d_2, c_i, c_j\}$ is an MC_4 in G for some $\{c_i, c_j\} \subseteq C \cap C_1\}$. This follows from Lemma 5.1. Let $C_1 = \{c_1, c_2, c_3, c_5\}$. Let $U = \{c_1, c_2, \ldots, c_5\}$, and F be the set of the edges $\{d_1, d_2\}$ in $G - U$ such that $\{d_1, d_2, c_i, c_j\}$ is an MC_4 in G for some $c_i \in U$ and $c_j \in U$. Then, every correct nonpizza layout of $G[U]$ can be transformed to another of the form in Figure 5.4.1. Figure 5.4.1 is explicit and (c'_1, c'_2, c'_3) is a permutation of (c_1, c_2, c_3) in it.

Similarly to Fact 5, we can prove the following fact:

Fact 8 For some $W \subseteq U$, $\mathcal{C}^G_{U,F}(W \cup \{c_4, c_5\}) \neq \emptyset$.

Assume that $G[C]$ has a correct nonpizza layout. We want to find a permutation (c'_1, c'_2, c'_3) of (c_1, c_2, c_3) such that the layout in Figure 5.4.1 is correct. To this end, let F_4 be the set of the edges $\{d_1, d_2\}$ in $G - C$ such that $\{d_1, d_2, c_i, c_j\}$ is an MC_4 in G for some $\{c_i, c_j\} \subseteq C$. Let K_5 be the connected component of the graph $G - C - F_4$ containing c_5. By Fact 8, $c_4 \in N_G(K_5)$. We distinguish three cases according to the value of $|N_G(K_5 - \{c_5\}) \cap \{c_1, c_2, c_3\}|$ as follows.

Case 1: $|N_G(K_5 - \{c_5\}) \cap \{c_1, c_2, c_3\}| = 2$. We assume that $\{c_1, c_2\} \subseteq N_G(K_5 - \{c_5\})$; other cases are similar. Then, setting $(c'_1, c'_2, c'_3) = (c_1, c_2, c_3)$ in Figure 5.4.1 leads to a correct layout of $G[U]$.

Case 2: $|N_G(K_5 - \{c_5\}) \cap \{c_1, c_2, c_3\}| = 0$. Then, for every permutation (c'_1, c'_2, c'_3) of (c_1, c_2, c_3), the layout in Figure 5.4.1 is correct iff the family $\{\mathcal{C}^G_{C,F_4}(W) \mid W \in \mathcal{W}_a\}$ is a partition of $V(G) - (C \cup K_5)$, where $\mathcal{W}$ is the family of the sets $\{c_1, c_4\}$, $\{c_2, c_4\}$, $\{c_3, c_4\}$, $\{c'_1, c'_3\}$, $\{c'_2, c'_3\}$, $\{c'_1, c'_3, c_4\}$, and $\{c'_2, c'_3, c_4\}$. This can be shown by considering the maximum extension of the layout in Figure 5.4.1.

Case 3: $|N_G(K_5 - \{c_5\}) \cap \{c_1, c_2, c_3\}| = 1$. Suppose that $c_1 \in N_G(K_5 - \{c_5\})$; other cases are similar. Then, we may assume that $c'_1 = c_1$ in Figure 5.4.1. By considering the maximum extension of the layout of $G[U]$ in Figure 5.4.1, we can prove the following: For every permutation (c'_2, c'_3) of (c_2, c_3), the layout of $G[U]$ in Figure 5.4.1 is correct iff the following (1) through (5) hold :

(1) The family $\mathcal{F} = \{K_5\} \cup \{\mathcal{C}^G_{C,F_4}(W) \mid W \in \mathcal{W}\}$ is a partition of $V(G) - C$, where $\mathcal{W}$ is the family of the following sets: $\{c_1, c_4\}$, $\{c'_2, c_4\}$, $\{c'_3, c_4\}$, $\{c_1, c'_3\}$, $\{c'_2, c'_3\}$, $\{c_1, c'_2, c_4\}$, $\{c_1, c'_3, c_4\}$, and $\{c'_2, c'_3, c_4\}$.

(2) Let G' be the graph $(\mathcal{F}, E')$, where E' consists of the edges $\{X, Y\}$ such that $X \in \mathcal{F}$, $Y \in \mathcal{F}$, and $X \cap N_G(Y) \neq \emptyset$. If $\mathcal{C}^G_{C,F_4}(\{c_1, c'_2, c_4\}) = \emptyset$, then G' is a subgraph of the graph G_a in Figure 5.4.2(a); otherwise, $N_G(K_5) \cap \mathcal{C}^G_{C,F_4}(\{c_1, c'_2, c_4\}) \neq \emptyset$ and G' is a subgraph of the graph G_b in Figure 5.4.2(b).

(3) For each $W \in \mathcal{W}$ such that $|W| = 2$ and $deg_{G_a}(\mathcal{C}^G_{C,F_4}(W)) = 2$, the edge between the two neighbors of $\mathcal{C}^G_{C,F_4}(W)$ in G_a is also in G' only if $\mathcal{C}^G_{C,F_4}(W) = \emptyset$. Similarly, for each $W \in \mathcal{W}$ such that $|W| = 2$ and $deg_{G_b}(\mathcal{C}^G_{C,F_4}(W)) = 2$, the edge between the two neighbors of $\mathcal{C}^G_{C,F_4}(W)$ in G_b is also in G' only if $\mathcal{C}^G_{C,F_4}(W) = \emptyset$.

(4) For each edge $\{X, Y\} = \{\mathcal{C}^G_{C,F_4}(W_1), \mathcal{C}^G_{C,F_4}(W_2)\}$ in G', $|N_G(X) \cap Y| = 1$, $|X \cap N_G(Y)| = 1$, and $(W_1 \cap W_2) \cup (N_G(X) \cap Y) \cup (X \cap N_G(Y))$ is an MC_4 in G and is the unique maximal clique in G containing the two countries in $(X \cap N_G(Y)) \cup (N_G(X) \cap Y)$.

(5) If K_5 has a neighbor X in G', then $|N_G(X) \cap K_5| = 1$, $|X \cap N_G(K_5)| = 1$, and $\{c_1, c_4\} \cup (N_G(X) \cap K_5) \cup (X \cap N_G(K_5))$ is an MC_4 in G and is the unique maximal clique in G containing the two countries in $(X \cap N_G(K_5)) \cup (N_G(X) \cap K_5)$.

By the above discussions, we know how to compute a permutation (c'_1, c'_2, c'_3) of (c_1, c_2, c_3) such that the layout of $G[U]$ in Figure 5.4.1 is correct under the assumption that $G[C]$ has a correct nonpizza layout. So, we may assume that $(c'_1, c'_2, c'_3) = (c_1, c_2, c_3)$ in Figure 5.4.1. Let $\mathcal{L}_{np}$ be the layout in Figure 5.4.1. Our next task is to decide whether $\mathcal{L}_{np}$ is correct or $G[C]$ has no correct nonpizza layout.

Proposition 5.12 Let U' be an arbitrary set of vertices in G such that $U \subseteq U'$. If $G[U']$ has a correct layout from which deleting the countries of $U' - U$ yields the layout of $G[U]$ shown in Figure 5.4.0, then $\mathcal{L}_{np}$ is correct.

Proof. It suffices to observe that the layout in Figure 5.4.0 can be transformed to $\mathcal{L}_{np}$. ∎

The following (A_1) is a necessary condition for $\mathcal{L}_{np}$ to be correct:

(A_1) $\mathcal{C}^G_{U,F}(\{c_3, c_5\} \cup W) = \emptyset$ for every $W \subset U$.

We may assume that (A_1) holds; otherwise, $G[C]$ has no correct nonpizza layout.

Lemma 5.13 Suppose that there is no $d \notin U$ such that $\{c_1, c_2, c_5, d\}$ is an MC_4 in G. Let $X = \mathcal{C}^G_{U,F}(\{c_1, c_2, c_5\})$. For each $c_i \in \{c_1, c_2\}$, let $Y_i = \mathcal{C}^G_{U,F}(\{c_i, c_3, c_4\})$ and $Z_i = \mathcal{C}^G_{U,F}(\{c_i, c_4, c_5\})$. Then, $\mathcal{L}_{np}$ is correct iff the following (1) through (4) hold:

(1) $\mathcal{C}^G_{U,F}(\{c_1, c_2\}) = \emptyset$.

(2) If $\mathcal{C}^G_{U,F}(\{c_4, c_5\}) \neq \emptyset$, then $X = \emptyset$.

(3) For each $c_i \in \{c_1, c_2\}$, if $Y_i \cap N_G(Z_i) \neq \emptyset$, then $\mathcal{C}^G_{U,F}(\{c_i, c_4\}) = \emptyset$.

(4) For each $c_i \in \{c_1, c_2\}$, if $X \neq \emptyset$ and $Z_i \neq \emptyset$, then $|N_G(X) \cap Z_i| = 1$, $|X \cap N_G(Z_i)| = 1$, $\{c_i, c_5\} \cup (N_G(X) \cap Z_i) \cup (X \cap N_G(Z_i))$ is an MC_4 in G, $X \cap N_G(c_i) = X \cap N_G(Z_i)$, and $N_G(c_5) \cap Z_i = N_G(X) \cap Z_i$.

Proof. The necessary condition follows from Fact 8 and Figure 5.4.1 immediately. To see the sufficient condition, suppose that (1) through (4) in the lemma hold. Then, every correct layout of $G[U]$ in which $G[C]$ is layouted as a pizza can be transformed to one of the three layouts in Figure 5.4.3. All the three figures in Figure 5.4.3 are explicit. By Proposition 5.12, the layout in Figure 5.4.3(a) can be transformed to $\mathcal{L}_{np}$. It remains to show that the last two layouts in Figure 5.4.3 can be transformed to $\mathcal{L}_{np}$. We only prove that the layout in Figure 5.4.3(b) can be transformed to $\mathcal{L}_{np}$; the other case is similar. Assume that the layout in Figure 5.4.3(b) is correct. Our goal is to show that $\mathcal{L}_{np}$ is also correct. By (A_1), (1) and Figure 5.4.3(b), the family $\{\mathcal{C}^G_{U,F}(W) \mid W \in \mathcal{W}\}$ is a partition of $V(G) - U$, where $\mathcal{W}$ is the class of the following sets: $\{c_1, c_3\}$, $\{c_1, c_4\}$, $\{c_1, c_5\}$, $\{c_2, c_3\}$, $\{c_2, c_5\}$, $\{c_3, c_4\}$, $\{c_4, c_5\}$, $\{c_1, c_2, c_5\}$, $\{c_1, c_3, c_4\}$, and $\{c_1, c_4, c_5\}$. Using this fact and (1) through (4) in the lemma, it is easy to prove that the layout in Figure 5.4.3(b) can be transformed to $\mathcal{L}_{np}$. ∎

By Lemma 5.13, we may assume that there is a country $c_6 \notin U$ such that $\{c_1, c_2, c_5, c_6\}$ is an MC_4 in G. Let $U_6 = U \cup \{c_6\}$, and F_6 be the set of the edges $\{d_1, d_2\}$ in $G - U_6$ such that $\{d_1, d_2, c_i, c_j\}$ is an MC_4 in G for some $\{c_i, c_j\} \subseteq U_6$.

Lemma 5.14 Suppose that $\{c_6, c_4\} \notin E(G)$. Then, $\mathcal{L}_{np}$ is correct iff the following (1) through (4) hold:

(1) $c_6 \notin \mathcal{C}^G_{U,F}(\{c_1, c_2, c_5\})$.

(2) For every $W \subset U_6$, $\mathcal{C}^G_{U_6,F_6}(\{c_3, c_5\} \cup W) = \emptyset$, $\mathcal{C}^G_{U_6,F_6}(\{c_3, c_6\} \cup W) = \emptyset$, and $\mathcal{C}^G_{U_6,F_6}(\{c_4, c_5\} \cup W) = \emptyset$.

(3) $\mathcal{C}^G_{U_6,F_6}(\{c_1, c_2\}) = \mathcal{C}^G_{U_6,F_6}(\{c_1, c_2, c_5\}) = \emptyset$.

(4) For each $c_i \in \{c_1, c_2\}$, (i) $\mathcal{C}^G_{U_6,F_6}(\{c_i, c_6\}) = \emptyset$ if $N_G(\mathcal{C}^G_{U_6,F_6}(\{c_i, c_5, c_6\})) \cap \mathcal{C}^G_{U_6,F_6}(\{c_i, c_4, c_6\}) \neq \emptyset$, and (ii) $\mathcal{C}^G_{U_6,F_6}(\{c_i, c_4\}) = \emptyset$ if $N_G(\mathcal{C}^G_{U_6,F_6}(\{c_i, c_4, c_6\})) \cap \mathcal{C}^G_{U_6,F_6}(\{c_i, c_4, c_3\}) \neq \emptyset$.

Proof. ($\Longrightarrow$) Assume that $\mathcal{L}_{np}$ is correct. Then, the MC$_4$ $C_2 = \{c_1, c_2, c_5, c_6\}$ has a correct nonpizza layout. Thus, the best nonpizza layout of $G[C_2]$ is of 1-type. This implies that every correct layout of $G[U_6]$ in which $G[C]$ is layouted as a nonpizza can be transformed to the one in Figure 5.4.4. Figure 5.4.4 is explicit. ¿From this figure, it is easy to see (1) through (4) in the lemma hold.

($\Longleftarrow$) Assume that (1) through (4) in the lemma hold. Then, every correct layout of $G[U_6]$ in which $G[C]$ is layouted as a pizza can be transformed to one of the three layouts in Figure 5.4.5. Figure 5.4.5(a) becomes explicit after c_6 is deleted from it. Both Figure 5.4.5(b) and Figure 5.4.5(c) are explicit. By Proposition 5.12, the layout in Figure 5.4.5(a) can be transformed to $\mathcal{L}_{np}$. It remains to show that the last two layouts in Figure 5.4.5 can be transformed to $\mathcal{L}_{np}$. We only prove that the layout in Figure 5.4.5(b) can be transformed to $\mathcal{L}_{np}$; the other case is similar. Assume that the layout in Figure 5.4.5(b) is correct. Our goal is to show that $\mathcal{L}_{np}$ is also correct. By (1) and Figure 5.4.5(b), the family $\{\mathcal{C}^G_{U_6,F_6}(W) \mid W \in \mathcal{W}\}$ is a partition of $V(G) - U_6$, where $\mathcal{W}$ is the class of the following sets: $\{c_1, c_3\}$, $\{c_1, c_4\}$, $\{c_1, c_5\}$, $\{c_1, c_6\}$, $\{c_2, c_3\}$, $\{c_2, c_5\}$, $\{c_3, c_4\}$, $\{c_4, c_6\}$, $\{c_5, c_6\}$, $\{c_1, c_3, c_4\}$, $\{c_1, c_4, c_6\}$, and $\{c_1, c_5, c_6\}$. Using this fact and (1) through (4) in the lemma, it is easy to prove that the layout in Figure 5.4.5(b) can be transformed to the one in Figure 5.4.4. ∎

By Lemma 5.14, we may assume that $\{c_4, c_6\} \in E(G)$. Then, $C_3 = \{c_1, c_2, c_4, c_6\}$ is an MC$_4$ in G. If $\mathcal{L}_{np}$ is correct, then $G[C_3]$ has a correct nonpizza layout. Thus, we may assume that the best nonpizza layout of $G[C_3]$ is of 1-type. Then, exactly one of the following (A_2) and (A_3) holds:

(A_2) There is a country $c_7 \notin U_6$ such that $\{c_1, c_4, c_6, c_7\}$ is an MC$_4$ in G and c_7 and c_2 belong to the same connected component in the graph $G' = G - \{c_1, c_4, c_6\} - \{\{d_1, d_2\} \mid \{d_1, d_2, c_i, c_j\}$ is an MC$_4$ in G for some $\{c_i, c_j\} \subseteq \{c_1, c_4, c_6\}\}$.

(A_3) There is a country $c_7 \notin U_6$ such that $\{c_2, c_4, c_6, c_7\}$ is an MC$_4$ in G and c_7 and c_1 belong to the same connected component in the graph $G' = G - \{c_2, c_4, c_6\} - \{\{d_1, d_2\} \mid \{d_1, d_2, c_i, c_j\}$ is an MC$_4$ in G for some $\{c_i, c_j\} \subseteq \{c_2, c_4, c_6\}\}$.

We assume that (A_2) holds; the other case is similar. Then by (A_1), it is not difficult to see that when the following (A_4) or (A_5) does not hold, every correct layout of $G[U]$ in which $G[C]$ is layouted as a pizza can be transformed to the one in Figure 5.4.0.

(A_4) There is no $d \notin U_6 \cup \{c_7\}$ such that $\{c_2, c_5, c_6, d\}$ or $\{c_1, c_5, c_6, d\}$ is an MC$_4$ in G.

(A_5) In the graph $G - \{c_1, c_3, c_4, c_5, c_6\}$, c_7 cannot reach c_2.

Thus, by Proposition 5.12, we may assume that both (A_4) and (A_5) hold. Then, every correct layout of $G[U_6]$ in which $G[C]$ is layouted as a nonpizza can be transformed to one of the two layouts in Figure 5.4.6. Figure 5.4.6(a) is explicit except that possibly, c_7 may touch c_5. Figure 5.4.6(b) is explicit except that possibly, c_7 may touch c_3. By the two layouts, a necessary condition for $\mathcal{L}_{np}$ to be correct is the following (A_6):

(A_6) There is exactly one $c_i \in \{c_3, c_5\}$ such that $c_7 \in \mathcal{C}^G_{U_6,F_6}(\{c_1, c_4, c_6, c_i\}$.

We may assume that (A_6) holds; otherwise, $\mathcal{L}_{np}$ is incorrect.

Lemma 5.15 Suppose that $c_7 \in \mathcal{C}^G_{U_6,F_6}(\{c_1, c_4, c_6, c_5\}$. Let K_7 be the connected component in $G - U_6 - F_6$ containing c_7. Then, $\mathcal{L}_{np}$ is correct iff the following (1) and (2) hold:

(1) The family $\mathcal{F} = \{\mathcal{C}^G_{U_6,F_6}(W) \mid W \in \mathcal{W}\}$ is a partition of $V(G) - (U_6 \cup K_7)$, where $\mathcal{W}$ is the class of the following sets: $\{c_1, c_3\}$, $\{c_1, c_4\}$, $\{c_1, c_5\}$, $\{c_2, c_3\}$, $\{c_2, c_4\}$, $\{c_2, c_5\}$, $\{c_2, c_6\}$, $\{c_3, c_4\}$, $\{c_4, c_6\}$, $\{c_5, c_6\}$, $\{c_1, c_3, c_4\}$, $\{c_2, c_3, c_4\}$, $\{c_2, c_4, c_6\}$, and $\{c_2, c_5, c_6\}$.

(2) $N_G(c_4) \cap K_7 = \{c_7\}$.

Proof. The necessary condition follows from Figure 5.4.6(a). To see the sufficient condition, assume (1) and (2) in the lemma hold. Then, every correct layout of $G[U_6]$ in which $G[C]$ is layouted as a pizza can be transformed to the one in Figure 5.4.7(a) or the one in Figure 5.4.7(b). Figure 5.4.7(a) becomes explicit after c_6 is deleted from it. Figure 5.4.7(b) is explicit. By Proposition 5.12, the layout in Figure 5.4.7(a) can be transformed to the layout in Figure 5.4.6(a). Using (1) and (2), it is easy to prove that the layout in Figure 5.4.7(b) can be transformed to the one in Figure 5.4.6(a). ∎

Lemma 5.16 Suppose that $c_7 \in \mathcal{C}^G_{U_6,F_6}(\{c_1, c_4, c_6, c_3\}$. Let K_7 be the connected component in $G - U_6 - F_6$ containing c_7. Then, $\mathcal{L}_{np}$ is correct iff the following (1) and (2) hold:

(1) The family $\mathcal{F} = \{\mathcal{C}^G_{U_6,F_6}(W) \mid W \in \mathcal{W}\}$ is a partition of $V(G) - (U_6 \cup K_7)$, where $\mathcal{W}$ is the class of the following sets: $\{c_1, c_3\}$, $\{c_1, c_5\}$, $\{c_1, c_6\}$, $\{c_2, c_3\}$, $\{c_2, c_4\}$, $\{c_2, c_5\}$, $\{c_2, c_6\}$, $\{c_3, c_4\}$, $\{c_4, c_6\}$, $\{c_5, c_6\}$, $\{c_1, c_5, c_6\}$, $\{c_2, c_3, c_4\}$, $\{c_2, c_4, c_6\}$, and $\{c_2, c_5, c_6\}$.

(2) $N_G(c_6) \cap K_7 = \{c_7\}$.

Proof. The necessary condition follows from Figure 5.4.6(b). On the other hand, if (1) and (2) in the lemma hold, then every correct layout of $G[U_6]$ in which $G[C]$ is layouted as a pizza can be transformed to the one in Figure 5.4.8. It is easy to see that when (1) and (2) in the lemma hold, the layout in Figure 5.4.8 can be transformed to the one in Figure 5.4.6(b). ∎