36th VTRMC, 2014, Solutions

1. Let $S$ denote the sum of the given series. By partial fractions,

$$ \frac{2n^2 - 2n - 4}{n^4 + 4n^2 + 16} = \frac{n - 2}{n^2 - 2n + 4} - \frac{n}{n^2 + 2n + 4}. $$

If $f(n) = \frac{n - 2}{n^2 - 2n + 4}$, then $2S = \sum_{n=2}^{\infty} f(n) - f(n+2)$. Since $\lim_{n \to \infty} f(n) = 0$, it follows by telescoping series that the series is convergent and $2S = f(2) - f(4) + f(3) - f(5) + f(4) - f(6) + \cdots$, so $2S = f(2) + f(3)$ and we deduce that $S = \frac{1}{14}$.

2. Let $I$ denote the given integral. First we make the substitution $y = x^2$, so $dy = 2xdx$. Then

$$ 2I = \int_0^4 \frac{16 - y}{16 - y + \sqrt{(16 - y)(12 + y)}} dy = \int_0^4 \frac{\sqrt{16 - y}}{\sqrt{16 - y + \sqrt{12 + y}}} dy. $$

Now make the substitution $z = 4 - y$, so $dz = -dy$. Then

$$ 2I = \int_0^4 \frac{\sqrt{12 + z}}{\sqrt{12 + z + \sqrt{16 - z}}} dz. $$

Adding the last two equations, we obtain $4I = \int_0^4 dz = 4$ and hence $I = 1$.

3. Let $m = \phi(2^{2014}) = 2^{2013}$ (here $\phi(x)$ is Euler’s totient function, the number of positive integers less than $x$ which are prime to $x$). Then $19^m \equiv 1 \mod 2^{2014}$ by Euler’s theorem. Therefore $n$ divides $2^{2013}$, so $n = 2^k$ for some positive integer $k$. Now

$$ 19^{2^k} - 1 = (19 - 1)(19 + 1)(19^2 + 1)(19^4 + 1)\cdots(19^{2^{k-1}} + 1); $$

we calculate the power of 2 in the above expression. This is $1 + 2 + 1 + \cdots + 1 = k + 2$. Therefore $k + 2 = 2014$ and it follows that $n = 2^{2012}$.

4. Put $i^{a+2b}$ in the square in the $(a, b)$ position. Note that the sum of all the entries in a $4 \times 1$ or $1 \times 4$ rectangle is zero, because $\sum_{k=0}^{3} i^{a+k+2b} = (1 + i + i^2 + i^3)i^{a+2b} = 0$ and $\sum_{k=0}^{3} i^{a+2(b+k)} = (1 + i^2 + i^4 + i^6)i^{a+2b} = 0$. Therefore if we have a tiling with $4 \times 1$ and $1 \times 4$ rectangles, the sum of the entries in
all 361 squares is the value on the central square, namely \(i^{10+20} = -1\). On
the other hand this sum is also
\[
(i^2 + \cdots + i^{19})(i^2 + i^4 + \cdots + i^{38}) = i^{19} \frac{1}{i-1} \cdot (-1 + 1 - \cdots - 1)
\]
\[
= i^{-1} \cdot -1 = 1.
\]
This is a contradiction and therefore we have no such tiling.

5. Suppose by way of contradiction we can write \(n(n+1)(n+2) = m^r\), where
\(n \in \mathbb{N}\) and \(r \geq 2\). If a prime \(p\) divides \(n(n+2)\) and \(n+1\), then it would have
to divide \(n+1\), and \(n\) or \(n+2\), which is not possible. Therefore we may
write \(n(n+2) = x'^{r}\) and \(n+1 = y^r\) for some \(x, y \in \mathbb{N}\). But then \(n(n+2)+1 =
(n+1)^2 = z^r\) where \(z = y^2\). Since \((n+1)^2 > n(n+2)\), we see that \(z > x\) and
hence \(z \geq x+1\), because \(x, z \in \mathbb{N}\). We deduce that \(z^r \geq (x+1)^r > x'^r + 1\), a
contradiction and the result follows.

6. (a) Since \(A\) and \(B\) are finite subsets of \(T\), we may choose \(a \in A\) and \(b \in B\)
so that \(f(ab)\) is as large as possible. Suppose we can write \(g := ab = cd\)
with \(c \in A\) and \(d \in B\). Let \(h = d^{-1}b\) and \(d \neq b\). Note that \(g, h \in T\).
Then \(h \neq I\) and we see that either \(f(gh^{-1}) > f(g)\) or \(f(gh) > f(g)\). This
contradicts the maximality of \(f(ab)\). Therefore \(d = b\) and because \(b\) is
an invertible matrix, we deduce that \(a = c\) and the result is proven.

(b) Set \(M = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}\). Then \(M \in S\) and \(M^3 = I\). Suppose \(f(M) > f(I)\).
Then \((X = M\) and \(Y = M)\) we obtain either \(f(M^2) > f(M)\) or \(f(I) >
f(M)\), hence \(f(M^2) > f(M)\). Now do the same with \(X = M^2\) and \(Y =
M\): we obtain \(f(M^3) > f(M^2)\). Since \(M^3 = I\), we now have \(f(I) >
f(M^2) > f(M) > f(I)\), a contradiction. The argument is similar if we
start out with \(f(M) < f(I)\). This shows that there is no such \(f\).

7. (a) Let \(A = (x_A, y_A)\) and \(B = (x_B, y_B)\). Then \(d(A, B) = \begin{pmatrix} x_B - x_A + y_A - y_B \\ x_B - x_A \end{pmatrix}\).

(b) By definition \(\det M = d(A_1, B_1)d(A_2, B_2) - d(A_1, B_2)d(A_2, B_1)\). Note
that the first term counts all pairs of paths \((\pi_1, \pi_2)\) where \(\pi_i : A_i \to B_i\),
and the second term is the negative of the number of pairs \((\pi_1, \pi_2)\) where
\(\pi_1 : A_1 \to B_2\) and \(\pi_2 : A_2 \to B_1\). The configuration of the points implies
that every pair of paths \((\pi_1, \pi_2)\) where \(\pi_1 : A_1 \to B_2\) and \(\pi_2 : A_2 \to B_1\)
must intersect. Let $\mathcal{I} := \{(\pi_1, \pi_2) : \pi_1 \cap \pi_2 \neq \emptyset\}$ (this is the set of all intersecting paths, regardless of their endpoints). Define $\Phi : \mathcal{I} \rightarrow \mathcal{I}$ as follows. If $(\pi_1, \pi_2) \in \mathcal{I}$ then $\Phi((\pi_1, \pi_2)) = (\pi'_1, \pi'_2)$ and the new pair of paths is obtained from the old one by switching the tails of $\pi_1, \pi_2$ after their last intersection point. In particular, the pairs $(\pi_1, \pi_2)$ and $(\pi'_1, \pi'_2)$ must appear in different terms of $\det M$. But it is clear that $\Phi \circ \Phi = id_{\mathcal{I}}$, therefore $\Phi$ is an involution. This implies that all intersecting pairs of paths must cancel each other, and that the only pairs which contribute to the determinant are those from the set $\{(\pi_1, \pi_2) : \pi_1 \cap \pi_2 = \emptyset\}$. Since all the latter pairs can appear only with positive sign (in the first term of $\det M$), this finishes the solution. (In fact, we proved that $\det M = \#\{(\pi_1, \pi_2) : \pi_1 \cap \pi_2 = \emptyset\}$.)