4. Ceva’s theorem applied to the triangle $ABC$ shows that $\frac{AR}{BP} \cdot \frac{BP}{CQ} \cdot \frac{CQ}{QA} = 1$.

Since $RP$ bisects $\angle BRC$, we see that $\frac{BP}{PC} = \frac{BR}{RC}$. Therefore $\frac{AR}{RC} = \frac{AQ}{QC}$, consequently $\angle ARQ = \angle QRC$ and the result follows.

5. Let

\[
A = (2 + \sqrt{5})^{100}((1 + \sqrt{2})^{100} + (1 + \sqrt{2})^{-100})
\]
\[
B = (\sqrt{5} - 2)^{100}((1 + \sqrt{2})^{100} + (1 + \sqrt{2})^{-100})
\]
\[
C = (\sqrt{5} + 2)^{100} + (\sqrt{5} - 2)^{100}
\]
\[
D = (\sqrt{2} + 1)^{100} + (\sqrt{2} - 1)^{100}
\]

First note that $C$ and $D$ are integers; one way to see this is to use the binomial theorem. Also $\sqrt{5} - 2 < 1/4$, $\sqrt{5} + 2 > 2.5$ and $\sqrt{5} - 1 < 1$. Therefore $0 < B < (5/8)^{100} + (1/4)^{100} < 10^{-4}$. We conclude that there is a positive number $\varepsilon < 10^{-4}$ such that $A + \varepsilon$ is an integer, hence the third digit after the decimal point of the given expression $A$ is 9.

6. Suppose $\det(A^2 + B^2) = 0$. Then $A^2 + B^2$ is not invertible and hence there exists a nonzero $n \times 1$ matrix (column vector) $u$ with real entries such that $(A^2 + B^2)u = 0$. Then $u^tA^2u + u^tB^2u = 0$, where $u^t$ denotes the transpose of $u$, a $1 \times n$ matrix. Therefore $(Au)^t(Au) + (Bu)^t(Bu) = 0$ and we deduce that $u^tA = u^tB = 0$, consequently $u^t(AX + BY) = 0$. This shows that $\det(AX + BY) = 0$, a contradiction and the result follows.

7. We claim that $\frac{x^{1/\ln(\ln x)^2}}{(\ln x)^2} > (\ln x)^2$ for large $x$. Indeed by taking logs, we need $(\ln x)/(\ln(\ln x))^2 > 2\ln(\ln x)$, that is $\ln x > 2(\ln(\ln x))^3$. So by making the substitution $y = \ln x$, we want $y > 2(\ln y)^3$, which is true for $y$ large. It now follows that for large $n$,

\[
n^{-1/(\ln(n)\ln(n))} = \frac{1}{n n^{1/(\ln(n)\ln(n))}} < \frac{1}{n(\ln(n))^2}.
\]

However $\sum 1/(n(\ln(n))^2)$ is well known to be convergent, by using the integral test, and it now follows from the basic comparison test that the given series is also convergent.