Therefore we may assume in any infinite subset of people there is a person who knows an infinite number of people. So we can pick a person \( B_1 \) who knows infinitely many people \( T_1 \). Then we can pick a person \( B_2 \) in \( T_1 \) who knows an infinite number of people \( T_2 \) of \( T_1 \), because we are assuming in any infinite subset of people, there is somebody who knows infinitely many of them. Of course, \( B_1 \) and \( B_2 \) know all the people in \( T_1 \), now choose a person \( B_3 \) in \( T_2 \) who knows an infinite number of people in \( T_2 \). Then \( \{B_1,B_2,B_3\} \) know each other. Clearly we can continue this process indefinitely to obtain an arbitrarily large number of people who know each other.

**Remark** A simple application of the axiom of choice shows that we can find an infinite number of people in the party such that either they all know each other or they all don’t know each other.

7. Set \( b_n = 1 - a_{n+1}/a_n \). Let us suppose to the contrary that \( \sum |b_n| \) is convergent. Then \( \lim_{n \to \infty} b_n = 0 \), so may assume that \( |b_n| < 1/2 \) for all \( n \). Now

\[
a_{n+1} = a_1(1 - b_1)(1 - b_2) \ldots (1 - b_n),
\]

hence

\[
\ln(a_{n+1}) = \ln a_1 + \ln(1 - b_1) + \ln(1 - b_2) + \cdots + \ln(1 - b_n).
\]

Since \( \lim_{n \to \infty} a_n = 0 \), we see that \( \lim_{n \to \infty} \ln(a_n) = -\infty \). Now \( \ln(1 - b) \geq -2|b| \) for \( |b| < 1/2 \); one way to see this is to observe that \( 1/(1 - x) \leq 2 \) for \( 0 \leq x \leq 1/2 \) and then to integrate between 0 and \( |b| \). Therefore \( \lim_{n \to \infty} (-2|b_1| - \cdots - 2|b_n|) = -\infty \). This proves that \( \sum |b_n| \) is divergent and the result follows.