35th VTRMC, 2013, Solutions

1. Make the substitution $t = 2y$, so $dt = 2dy$. Then $I = \int_0^{x/2} 6\sqrt[4]{\frac{(1+\cos 2y)^2}{17-8\cos 2y}} dy = \int_0^{x/2} 3\sqrt{2} \frac{2\sqrt{2}\cos y}{9+16\sin^2 y} dy$. Now make the substitution $z = \sin y$. Then $dz = dy \cos y$ and $I = 12 \int_0^{\sin x/2} \frac{dz}{3\sqrt{2} + 4\sqrt{2} \sin x/2} = \tan^{-1} \frac{1}{3} \sin x/2$. If $\tan I = 2/\sqrt{3}$, then $2\sqrt{3} = 4 \sin x/2$ and we deduce that $x = 2\pi/3$.

2. Without loss of generality we may assume that $BC = 1$, and then we set $x := BD$, so $AD = 2x$. Write $\theta = \angle CAD$, $y = AC$ and $z = DC$. The area of $ADC$ is both $x$ and $(yz \sin \theta)/2$. Also $y^2 = 9x^2$ and $z^2 = 1+x^2$. Therefore $4x^2 = (1+9x^2)(1+x^2) \sin^2 \theta$. We need to maximize $\theta$, equivalently $\sin^2 \theta$, which in turn is equivalent to minimizing $(1+9x^2)(1+x^2)/(4x^2)$. Therefore we need to find $x$ such that $x^2 + 9x^2$ is minimal. Differentiating, we find $-2x^3 + 18x = 0$, so $x^2 = 1/3$. It follows that $\sin^2 \theta = 1/4$ and we deduce that the maximum value of $\angle CAD = \theta$ is $30^\circ$.

3. We need to show that $a_n$ is bounded, equivalently $\ln a_n$ is bounded, i.e. $\ln 2 \sum_{n=1}^{\infty} \ln(1+n^{-3/2})$ is bounded. But $\ln(1+n^{-3/2}) < n^{-3/2}$ and $\sum_{n=1}^{\infty} n^{-3/2}$ is convergent. It follows that $(a_n)$ is convergent.

4. (a) $25 = 50/2 = \frac{5^2 + 5^2}{1^2 + 1^2}$.

(b) Assume that 2013 is special. Then we have

$$x^2 + y^2 = 2013(u^2 + v^2)$$ (1)

for some positive integers $x, y, u, v$. Also, we assume that $x^2 + y^2$ is minimal with this property. The prime factorization of 2013 is $3 \cdot 11 \cdot 61$. From (1) it follows $3|x^2 + y^2$. It is easy to check by looking to the residues mod 3 that $3|x$ and $3|y$, hence we have $x = 3x_1$ and $y = 3y_1$. Replacing in (1) we get

$$3(x_1^2 + y_1^2) = 11 \cdot 61(u^2 + v^2),$$ (2)

i.e. $3|u^2 + v^2$. It follows $u = 3u_1$ and $v = 3v_1$, and replacing in (2) we get

$$x_1^2 + y_1^2 = 2013(u_1^2 + v_1^2).$$

Clearly, $x_1^2 + y_1^2 < x^2 + y^2$, contradicting the minimality of $x^2 + y^2$. 