34th Annual Virginia Tech Regional Mathematics Contest
From 9:00 a.m. to 11:30 a.m., October 27, 2012

Fill out the individual registration form

1. Evaluate
\[ \int_{0}^{\pi/2} \frac{\cos^4 x + \sin x \cos^3 x + \sin^2 x \cos^2 x + \sin^3 x \cos x}{\sin^4 x + \cos^4 x + 2 \sin x \cos^3 x + 2 \sin^2 x \cos^2 x + 2 \sin^3 x \cos x} \, dx. \]

2. Solve in real numbers the equation \( 3x - x^3 = \sqrt{x + 2} \).

3. Find nonzero complex numbers \( a, b, c, d, e \) such that
   \[
   \begin{align*}
   a + b + c + d + e &= -1 \\
   a^2 + b^2 + c^2 + d^2 + e^2 &= 15 \\
   \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e} &= -1 \\
   \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{d^2} + \frac{1}{e^2} &= 15 \\
   abcde &= -1
   \end{align*}
   \]

4. Define \( f(n) \) for \( n \) a positive integer by \( f(1) = 3 \) and \( f(n + 1) = 3f(n) \). What are the last two digits of \( f(2012) \)?

5. Determine whether the series \( \sum_{n=1}^{\infty} \frac{1}{\ln n} - \left( \frac{1}{\ln n} \right)^{(n+1)/n} \) is convergent.

6. Define a sequence \( (a_n) \) for \( n \) a positive integer inductively by \( a_1 = 1 \) and \( a_n = \frac{n}{\prod_{d|n} a_d} \). Thus \( a_2 = 2, a_3 = 3, a_4 = 2 \) etc. Find \( a_{999000} \).

7. Let \( A_1, A_2, A_3 \) be \( 2 \times 2 \) matrices with entries in \( \mathbb{C} \) (the complex numbers). Let \( \text{tr} \) denote the trace of a matrix (so \( \text{tr} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + d \)). Suppose \( \{A_1, A_2, A_3\} \) is closed under matrix multiplication (i.e. given \( i, j \), there exists \( k \) such that \( A_iA_j = A_k \)), and \( \text{tr}(A_1 + A_2 + A_3) \neq 3 \). Prove that there exists \( i \) such that \( A_iA_j = A_jA_i \) for all \( j \) (here \( i, j \) are 1, 2 or 3).