33rd Annual Virginia Tech Regional Mathematics Contest  
From 9:00 a.m. to 11:30 a.m., October 29, 2011

Fill out the individual registration form

1. Evaluate \( \int_1^4 \frac{x-2}{(x^2+4)^{\frac{1}{2}}} \, dx \).

2. A sequence \((a_n)\) is defined by \(a_0 = -1, a_1 = 0,\) and
\[
a_{n+1} = a_n^2 - (n+1)^2a_{n-1} - 1
\]
for all positive integers \(n\). Find \(a_{100}\).

3. Find \(\sum_{k=1}^{\infty} \frac{k^2 - 2}{(k+2)!}\).

4. Let \(m, n\) be positive integers and let \([a]\) denote the residue class mod \(mn\) of the integer \(a\) (thus \([r] \mid r\) is an integer\) has exactly \(mn\) elements). Suppose the set \([ar] \mid r\) is an integer\) has exactly \(m\) elements. Prove that there is a positive integer \(q\) such that \(q\) is prime to \(mn\) and \([nq] = [a]\).

5. Find \(\lim_{x \to \infty} (2x)^{1+\frac{1}{x}} - x^{1+\frac{1}{x}} - x\).

6. Let \(S\) be a set with an asymmetric relation \(<\); this means that if \(a, b \in S\) and \(a < b\), then we do not have \(b < a\). Prove that there exists a set \(T\) containing \(S\) with an asymmetric relation \(<\) with the property that if \(a, b \in S\), then \(a < b\) if and only if \(a < b\), and if \(x, y \in T\) with \(x < y\), then there exists \(t \in T\) such that \(x < t < y\) (\(t \in T\) means “\(t\) is an element of \(T\)”).

7. Let \(P(x) = x^{100} + 20x^{99} + 198x^{98} + a_97x^{97} + \cdots + a_1x + 1\) be a polynomial where the \(a_i\) (\(1 \leq i \leq 97\)) are real numbers. Prove that the equation \(P(x) = 0\) has at least one complex root (i.e. a root of the form \(a + bi\) with \(a, b\) real numbers and \(b \neq 0\)).