Geometry Session

Today we look at some accessible “geometry” questions from recent Putnam exams. Most years there is at least one geometry question, and they are often among the easier ones, so it is worth getting some practice with them.

As usual we’ll take about half of our time just to look over the problems individually. Then we’ll discuss any ideas you might have, and how to write them up. After that you might look at the hints (second page). We may not have time to go over the solutions (3rd page).

1. PROBLEMS

Look over the problems below. Try to identify one or more problems where you have some idea of how to get started. For a real Putnam session, I recommend you spend at least half an hour just on this step!

**2013 A1:** Recall that a regular icosahedron is a convex polyhedron having 12 vertices and 20 faces; the faces are congruent equilateral triangles. On each face of a regular icosahedron is written a nonnegative integer such that the sum of all 20 integers is 39. Show that there are two faces that share a vertex and have the same integer written on them.

**2012 A1:** Let $d_1, d_2, \ldots, d_{12}$ be real numbers in the open interval $(1, 12)$. Show that there exist distinct indices $i, j, k$ such that $d_i, d_j, d_k$ are the side lengths of an acute triangle.

**2010 B2:** Given that $A, B,$ and $C$ are noncollinear points in the plane with integer coordinates such that the distances $AB, AC,$ and $BC$ are integers, what is the smallest possible value of $AB$?

**2008 B1:** What is the maximum number of rational points that can lie on a circle in $\mathbb{R}^2$ whose center is not a rational point? (A rational point is a point both of whose coordinates are rational numbers.)

**2004 A2:** For $i = 1, 2$ let $T_i$ be a triangle with side lengths $a_i, b_i, c_i,$ and area $A_i$. Suppose that $a_1 \leq a_2, b_1 \leq b_2, c_1 \leq c_2,$ and that $T_2$ is an acute triangle. Does it follow that $A_1 \leq A_2$?
II. HINTS

You won’t get hints on a real exam, but these ideas that may help you with similar problems might help. Look these over, and see if you can make any further progress.

2013 A1: Otherwise, you must have distinct integers around each vertex.

2012 A1: Look at the sequence of squares; show that if there is no acute triangle, the squares must “grow quickly”.

2010 B2: You know about the 3-4-5 triangle. Can you eliminate the smaller cases (0, 1, 2)?

2008 B1: Argue that two rational points determine a rational line, and vice versa.

2004 A2: First consider the special case $a_1 = a_2$.

The next page has solutions, don’t continue until you want to see them!
III. SOLUTIONS

2013 A1: Suppose otherwise. Then each vertex $v$ is a vertex for five faces, all of which have different labels, and so the sum of the labels of the five faces incident to $v$ is at least $0 + 1 + 2 + 3 + 4 = 10$. Adding this sum over all vertices $v$ gives $3 \times 39 = 117$, since each face’s label is counted three times. Since there are 12 vertices, we conclude that $10 \times 12 \leq 117$, contradiction.

2012 A1: Without loss of generality, assume $d_1 \leq d_2 \leq \ldots \leq d_{12}$. If $d_{i+2}^2 < d_i^2 + d_{i+1}^2$ for some $i \leq 10$, then $d_i, d_{i+1}, d_{i+2}$ are the side lengths of an acute triangle, since in this case $d_i^2 < d_{i+1}^2 + d_{i+2}^2$ and $d_{i+1}^2 < d_i^2 + d_{i+2}^2$ as well. Thus we may assume $d_{i+2}^2 \geq d_i^2 + d_{i+1}^2$ for all $i$. But then by induction, $d_i^2 \geq F_id_i^2$ for all $i$, where $F_i$ is the $i$-th Fibonacci number (with $F_1 = F_2 = 1$): $i = 1$ is clear, $i = 2$ follows from $d_2 \geq d_1$, and the induction step follows from the assumed inequality. Setting $i = 12$ now gives $d_{12}^2 \geq 144d_1^2$, contradicting $d_1 > 1$ and $d_{12} < 12$.

2010 B2: The smallest distance is 3, achieved by $A = (0, 0), B = (3, 0), C = (0, 4)$. To check this, it suffices to check that $AB$ cannot equal 1 or 2. (It cannot be 0 because the three points would be collinear.)

The triangle inequality implies that $|AC - BC| \leq AB$, with equality if and only if $A, B, C$ are collinear. If $AB = 1$, we may assume without loss of generality that $A = (0, 0), B = (1, 0)$. To avoid collinearity, we must have $AC = BC$, but this forces $C = (1/2, y)$ for some $y \in \mathbb{R}$, a contradiction. (One can also treat this case by scaling by a factor of 2 to reduce to the case $AB = 2$, treated in the next paragraph.)

If $AB = 2$, then we may assume without loss of generality that $A = (0, 0), B = (2, 0)$. The triangle inequality implies $|AC - BC| \in \{0, 1\}$. Also, for $C = (x, y), AC^2 = x^2 + y^2$ and $BC^2 = (2-x)^2 + y^2$ have the same parity; it follows that $AC = BC$. Hence $c = (1, y)$ for some $y \in \mathbb{R}$, so $y^2$ and $y^2 + 1 = BC^2$ are consecutive perfect squares. This can only happen for $y = 0$, but then $A, B, C$ are collinear, a contradiction again.

2008 B1: There are at most two such points. For example, the points $(0, 0)$ and $(1, 0)$ lie on a circle with center $(1/2, x)$ for any real number $x$, not necessarily rational.

On the other hand, suppose $P = (a, b), Q = (c, d), R = (e, f)$ are three rational points that lie on a circle. The midpoint $M$ of the side $PQ$ is $((a + c)/2, (b + d)/2)$, which is again rational. Moreover, the slope of the line $PQ$ is $(d - b)/(c - a)$, so the slope of the line through $P$ and $Q$ is $(a - c)/(b - d)$, which is rational or infinite.

Similarly, if $N$ is the midpoint of $QR$, then $N$ is a rational point and the line through $N$ perpendicular to $QR$ has rational slope. The center of the circle lies on both of these lines, so its coordinates $(g, h)$ satisfy two linear equations with rational coefficients, say $Ag + Bh = C$ and $Dg + Eh = F$. Moreover, these equations have a unique solution. That solution must then be

$$g = (CE - BD)/(AE - BD)$$
$$h = (AF - BC)/(AE - BD)$$

(by elementary algebra, or Cramer’s rule), so the center of the circle is rational. This proves the desired result.

2004 A2: Yes, it follows. For $i = 1, 2$, let $P_i, Q_i, R_i$ be the vertices of $T_i$ opposite the sides of length $a_i, b_i, c_i$, respectively.

We first check the case where $a_1 = a_2$ (or $b_1 = b_2$ or $c_1 = c_2$, by the same argument after relabeling). Imagine $T_2$ as being drawn with the base $Q_2R_2$ horizontal and the point $P_2$ above the line $Q_2R_2$. We may then position $T_1$ so that $Q_1 = Q_2$, $R_1 = R_2$, and $P_1$ lies above the line $Q_1R_1 = Q_2R_2$. Then $P_1$ also lies inside the region bounded by the circles through $P_2$ centered at $Q_2$ and $R_2$. Since $\angle Q_2$ and $\angle R_2$ are acute, the part of this region above the line $Q_2R_2$ lies within $T_2$. In particular, the distance from $P_1$ to the line $Q_2R_2$ is less than or equal to the distance from $P_2$ to the line $Q_2R_2$: hence $A_1 \leq A_2$.

To deduce the general case, put

$$r = \max\{a_1/a_2, b_1/b_2, c_1/c_2\}.$$

Let $T_3$ be the triangle with sides $ra_2, rb_2, rc_2$, which has area $r^2 A_2$. Applying the special case to $T_1$ and $T_3$, we deduce that $A_1 \leq r^2 A_2$; since $r \leq 1$ by hypothesis, we have $A_1 \leq A_2$ as desired.