Today we are looking at some of the easier problems from the 2003 Putnam. There is no intended theme today. As usual we’ll take about half our time looking at the problems. Then we’ll discuss any ideas you have, and I’ll offer hints on the remaining problems. Full solutions are on the third page, you can read them after our meeting.

The problem statements and solutions are from Kedlaya’s archive. (Often there are multiple solutions, I just pick one here.)

I. PROBLEMS

Look over the problems below. Try to identify one or more problems where you have some idea of how to get started. For a real Putnam session, I recommend you spend at least half an hour just on this step!

A–1: Let \( n \) be a fixed positive integer. How many ways are there to write \( n \) as a sum of positive integers, \( n = a_1 + a_2 + \cdots + a_k \), with \( k \) an arbitrary positive integer and \( a_1 \leq a_2 \leq \cdots \leq a_k \leq a_1 + 1 \)? For example, with \( n = 4 \) there are four ways: 4, 2+2, 1+1+2, 1+1+1+1.

A–2: Let \( a_1, a_2, \ldots, a_n \) and \( b_1, b_2, \ldots, b_n \) be nonnegative real numbers. Show that
\[
(a_1 a_2 \cdots a_n)^{1/n} + (b_1 b_2 \cdots b_n)^{1/n} \leq [(a_1 + b_1)(a_2 + b_2)\cdots(a_n + b_n)]^{1/n}.
\]

A–6: For a set \( S \) of nonnegative integers, let \( r_S(n) \) denote the number of ordered pairs \((s_1,s_2)\) such that \( s_1 \in S, s_2 \in S, s_1 \neq s_2 \), and \( s_1 + s_2 = n \). Is it possible to partition the nonnegative integers into two sets \( A \) and \( B \) in such a way that \( r_A(n) = r_B(n) \) for all \( n \)?

B–1: Do there exist polynomials \( a(x), b(x), c(y), d(y) \) such that
\[
1 + xy + x^2 y^2 = a(x)c(y) + b(x)d(y)
\]
holds identically?

B–2: Let \( n \) be a positive integer. Starting with the sequence 1, \( \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n} \), form a new sequence of \( n - 1 \) entries \( \frac{3}{4}, \frac{5}{12}, \ldots, \frac{2n-1}{2n(n-1)} \) by taking the averages of two consecutive entries in the first sequence. Repeat the averaging of neighbors on the second sequence to obtain a third sequence of \( n - 2 \) entries, and continue until the final sequence produced consists of a single number \( x_n \). Show that \( x_n < \frac{2}{n} \).

B–3: For each positive integer \( n \), show \( n! = \prod_{i=1}^{n} \text{lcm}\{1,2,\ldots,[n/i]\} \). (lcm is least common multiple, \( [x] \) is greatest integer \( \leq x \).)
II. HINTS

You won’t get hints on a real exam, but these ideas that may help you with similar problems might help. Look these over, and see if you can make any further progress.

A–1: Work out some small examples, find the pattern.

A–2: Apply the AM-GM inequality (the geometric mean of $n$ positive reals is at most their arithmetic mean).

A–6: Work out such a partition for the first few integers, try to spot a pattern (think in binary).

B–1: Think about linear independence of polynomials.

B–2: Work out some small examples, guess a formula for the general term, and prove it. (Maybe: think about random walks in a grid.)

B–3: Work out the exponent of a prime $p$, on both sides.

The next page has solutions, don’t continue until you want to see them!
III. SOLUTIONS

A–1: There are \( n \) such sums. More precisely, there is exactly one such sum with \( k \) terms for each of \( k = 1, \ldots, n \) (and clearly no others). To see this, note that if \( n = a_1 + a_2 + \cdots + a_k \) with \( a_1 \leq a_2 \leq \cdots \leq a_k \leq a_1 + 1 \), then

\[
ka_1 = a_1 + a_1 + \cdots + a_1 \leq n \leq a_1 + (a_1 + 1) + \cdots + (a_1 + 1) = ka_1 + k - 1.
\]

However, there is a unique integer \( a_1 \) satisfying these inequalities, namely \( a_1 = \lfloor n/k \rfloor \). Moreover, once \( a_1 \) is fixed, there are \( k \) different possibilities for the sum \( a_1 + a_2 + \cdots + a_k \): if \( i \) is the last integer such that \( a_i = a_1 \), then the sum equals \( ka_1 + (i - 1) \). The possible values of \( i \) are 1, \ldots, \( k \), and exactly one of these sums comes out equal to \( n \), proving our claim.

A–2: Assume that \( a_i + b_i > 0 \) for each \( i \) (otherwise both sides of the inequality are zero). Then the AM-GM inequality gives

\[
\left( \frac{a_1 \cdots a_n}{(a_1 + b_1) \cdots (a_n + b_n)} \right)^{1/n} \leq \frac{1}{n} \left( \frac{a_1}{a_1 + b_1} + \cdots + \frac{a_n}{a_n + b_n} \right),
\]

and likewise with \( a \) and \( b \) reversed. Adding these two inequalities and clearing denominators yields the result.

A–6: Yes, such a partition is possible. To achieve it, place each integer into \( A \) if it has an even number of 1s in its binary representation, and into \( B \) if it has an odd number. (One discovers this by simply attempting to place the first few numbers by hand and noticing the resulting pattern.)

To show that \( r_A(n) = r_B(n) \), we exhibit a bijection between the pairs \( (a_1, a_2) \) of distinct elements of \( A \) with \( a_1 + a_2 = n \) and the pairs \( (b_1, b_2) \) of distinct elements of \( B \) with \( b_1 + b_2 = n \). Namely, given a pair \( (a_1, a_2) \) with \( a_1 + a_2 = n \), write both numbers in binary and find the lowest-order place in which they differ (such a place exists because \( a_1 \neq a_2 \)). Change both numbers in that place and call the resulting numbers \( b_1, b_2 \). Then \( a_1 + a_2 = b_1 + b_2 = n \), but the parity of the number of 1s in \( b_1 \) is opposite that of \( a_1 \), and likewise between \( b_2 \) and \( a_2 \). This yields the desired bijection.

B–1: No. Suppose the contrary. By setting \( y = -1, 0, 1 \) in succession, we see that the polynomials \( 1 - x + x^2, 1, 1 + x + x^2 \) are linear combinations of \( a(x) \) and \( b(x) \). But these three polynomials are linearly independent, so they cannot all be written as linear combinations of two other polynomials, contradiction.

B–2: It is easy to see by induction that the \( j \)-th entry of the \( k \)-th sequence is \( \sum_{i=1}^{k} \binom{k-1}{i-1}/(2^{k-1}(i+j-1)) \), and so \( x_n = \frac{1}{2^n - 1} \sum_{i=1}^{n} \binom{n-1}{i-1}/i \). Now \( \binom{n-1}{i-1}/i = \binom{n}{i}/n \); hence \( x_n = \frac{1}{n2^n-1} \sum_{i=1}^{n} \binom{n}{i} = \frac{2^n-1}{n2^n-1} < 2/n \), as desired.

B–3: It is enough to show that for each prime \( p \), the exponent of \( p \) in the prime factorization of both sides is the same. On the left side, the exponent of \( p \) in the prime factorization of \( n! \) is \( \sum_{i=1}^{n} \lfloor n/p^i \rfloor \). (To see this, note that the \( i \)-th term counts the multiples of \( p^i \) among 1, \ldots, \( n \), so that a number divisible exactly by \( p^i \) gets counted exactly \( i \) times.) This number can be reinterpreted as the cardinality of the set \( S \) of points in the plane with positive integer coordinates lying on or under the curve \( y = np^{-x} \): namely, each summand is the number of points of \( S \) with \( x = i \).

On the right side, the exponent of \( p \) in the prime factorization of \( \text{lcm}(1, \ldots, \lfloor n/i \rfloor) \) is \( \lfloor \log_p(n/i) \rfloor = \lfloor \log_p(n/i) \rfloor \). This is precisely the number of points of \( S \) with \( y = i \). So the total exponent \( \sum_{i=1}^{n} \lfloor \log_p(n/i) \rfloor \) equals \( |S| \), and the result follows.