I. PROBLEMS

Look over the problems below. Try to identify one or more problems where you have some idea of how to get started. For a real Putnam session, I recommend you spend at least half an hour just on this step!

A–1: Let $k$ be a fixed positive integer. The $n$-th derivative of $\frac{1}{x^k-1}$ has the form $\frac{P_n(x)}{(x^k-1)^n}$ where $P_n(x)$ is a polynomial. Find $P_n(1)$.

A–2: Given any five points on a sphere, show that some four of them must lie on a closed hemisphere.

A–3: Let $n\geq 2$ be an integer and $T_n$ be the number of non-empty subsets $S$ of $\{1, 2, 3, \ldots, n\}$ with the property that the average of the elements of $S$ is an integer. Prove that $T_n - n$ is always even.

B–1: Shanille O’Keal shoots free throws on a basketball court. She hits the first and misses the second, and thereafter the probability that she hits the next shot is equal to the proportion of shots she has hit so far. What is the probability she hits exactly 50 of her first 100 shots?

B–2: Consider a polyhedron with at least five faces such that exactly three edges emerge from each of its vertices. Two players play the following game:

Each player, in turn, signs his or her name on a previously unsigned face. The winner is the player who first succeeds in signing three faces that share a common vertex.

Show that the player who signs first will always win by playing as well as possible.

B–3: Show that, for all integers $n > 1$,

$$\frac{1}{2ne} < \frac{1}{e} - \left(1 - \frac{1}{n}\right)^n < \frac{1}{ne}.$$
II. HINTS

You won’t get hints on a real exam, but these ideas that may help you with similar problems might help. Look these over, and see if you can make any further progress.

A–1: Differentiate, and induction.
A–2: Pigeon hole.
A–3: Try to “pair up” sets.

B–1: Work out small examples first.
B–2: (Note: two faces can share at most one edge.) What if a face has at least four sides?
B–3: Taylor series.

The next page has solutions, don’t continue until you want to see them!
III. SOLUTIONS

A–1: By differentiating $P_n(x)/(x^k - 1)^{n+1}$, we find that $P_{n+1}(x) = (x^k - 1)P_n'(x) - (n+1)x^{k-1}P_n(x)$; substituting $x = 1$ yields $P_{n+1}(1) = -(n+1)kP_n(1)$. Since $P_0(1) = 1$, an easy induction gives $P_n(1) = (-k)^n n!$ for all $n \geq 0$.

Note: one can also argue by expanding in Taylor series around 1.

A–2: Draw a great circle through two of the points. There are two closed hemispheres with this great circle as boundary, and each of the other three points lies in one of them. By the pigeonhole principle, two of those three points lie in the same hemisphere, and that hemisphere thus contains four of the five given points.

Note: by a similar argument, one can prove that among any $n + 3$ points on an $n$-dimensional sphere, some $n + 2$ of them lie on a closed hemisphere. (One cannot get by with only $n + 2$ points: put them at the vertices of a regular simplex.)

A–3: Note that each of the sets $\{1\}, \{2\}, \ldots, \{n\}$ has the desired property. Moreover, for each set $S$ with integer average $m$ that does not contain $m$, $S \cup \{m\}$ also has average $m$, while for each set $T$ of more than one element with integer average $m$ that contains $m$, $T \setminus \{m\}$ also has average $m$. Thus the subsets other than $\{1\}, \{2\}, \ldots, \{n\}$ can be grouped in pairs, so $T_n - n$ is even.

B–1: The probability is $1/99$. In fact, we show by induction on $n$ that after $n$ shots, the probability of having made any number of shots from 1 to $n - 1$ is equal to $1/(n - 1)$. This is evident for $n = 2$. Given the result for $n$, we see that the probability of making $i$ shots after $n + 1$ attempts is $\frac{i - 1}{n} - \frac{1}{n^2} + \frac{1}{n + 1} - \frac{1}{n + 1} = 1 - \frac{1}{n + 1} = \frac{1}{n} - \frac{1}{n + 1}$, as claimed.

B–2: (Note: the problem statement assumes that all polyhedra are connected and that no two edges share more than one face, so we will do likewise. In particular, these are true for all convex polyhedra.) We show that in fact the first player can win on the third move. Suppose the polyhedron has a face $A$ with at least four edges. If the first player plays there first, after the second player’s first move there will be three consecutive faces $B, C, D$ adjacent to $A$ which are all unoccupied. The first player wins by playing in $C$; after the second player’s second move, at least one of $B$ and $D$ remains unoccupied, and either is a winning move for the first player.

It remains to show that the polyhedron has a face with at least four edges. (Thanks to Russ Mann for suggesting the following argument.) Suppose on the contrary that each face has only three edges. Starting with any face $F_1$ with vertices $v_1, v_2, v_3$, let $v_4$ be the other endpoint of the third edge out of $v_1$. Then the faces adjacent to $F_1$ must have vertices $v_1, v_2, v_4; v_1, v_3, v_4; v_2, v_3, v_4$. Thus $v_1, v_2, v_3, v_4$ form a polyhedron by themselves, contradicting the fact that the given polyhedron is connected and has at least five vertices. (One can also deduce this using Euler’s formula $V - E + F = 2 - 2g$, where $V, E, F$ are the numbers of vertices, edges and faces, respectively, and $g$ is the genus of the polyhedron. For a convex polyhedron, $g = 0$ and you get the “usual” Euler’s formula.)

Note: there is a counterexample if one relaxes the assumption that a pair of faces may not share multiple edges.

B–3: The desired inequalities can be rewritten as $1 - \frac{1}{n} < \exp \left(1 + n \log \left(1 - \frac{1}{n}\right)\right) < 1 - \frac{1}{nt}$.

Taking logarithms, we can rewrite the desired inequalities as $-\log \left(1 - \frac{1}{nt}\right) < -1 - n \log \left(1 - \frac{1}{n}\right) < - \log \left(1 - \frac{1}{n}\right)$.

Rewriting in terms of the Taylor expansion of $-\log(1 - x)$, the result is equivalent to $\sum_{i=1}^{\infty} \frac{1}{(i+1)^2} < \sum_{i=1}^{\infty} \frac{1}{(i+1)^n} < \sum_{i=1}^{\infty} \frac{1}{in}$.

These are true because the inequalities hold term by term.