Practice Exam Session

In this second “practice exam” session, we look at the problems from Session A of the 2010 Putnam. Again we won’t have
time to solve them all, but we will work on the basic skills (picking problems, remembering your math, maybe some writing).
This should take about 50 minutes.

The first page has the problems, the second page has hints (don’t look yet!), and the third page has solutions (save those for
the end). Both the problem statements and solutions are taken from Kedlaya’s archive,

I. (10-15 MINUTES): LOOK OVER THE PROBLEMS

Look over the problems below. Do not attempt to write out solutions, although do jot down basic ideas if you have them. Try
to identify one or more problems where you have some idea of some relevant mathematics, or some idea how to proceed. For a
real exam, you might give half an hour to just this step!

A–1 Given a positive integer \(n\), what is the largest \(k\) such that the numbers 1, 2, \ldots, \(n\) can be put into \(k\) boxes so that the sum
of the numbers in each box is the same? [When \(n = 8\), the example \{1, 2, 3, 6\}, \{4, 8\}, \{5, 7\} shows that the largest \(k\) is at
least 3.]

A–2 Find all differentiable functions \(f : \mathbb{R} \rightarrow \mathbb{R}\) such that

\[
f'(x) = \frac{f(x + n) - f(x)}{n}
\]

for all real numbers \(x\) and all positive integers \(n\).

A–3 Suppose that the function \(h : \mathbb{R}^2 \rightarrow \mathbb{R}\) has continuous partial derivatives and satisfies the equation

\[
h(x, y) = a \frac{\partial h}{\partial x}(x, y) + b \frac{\partial h}{\partial y}(x, y)
\]

for some constants \(a, b\). Prove that if there is a constant \(M\) such that \(|h(x, y)| \leq M\) for all \((x, y) \in \mathbb{R}^2\), then \(h\) is identically
zero.

A–4 Prove that for each positive integer \(n\), the number \(10^{10^0} + 10^{10^n} + 10^n - 1\) is not prime.

A–5 Let \(G\) be a group, with operation \(*\). Suppose that

(i) \(G\) is a subset of \(\mathbb{R}^3\) (but \(*\) need not be related to addition of vectors);

(ii) For each \(a, b \in G\), either \(a \times b = a * b\) or \(a \times b = 0\) (or both), where \(\times\) is the usual cross product in \(\mathbb{R}^3\).

Prove that \(a \times b = 0\) for all \(a, b \in G\).

A–6 Let \(f : [0, \infty) \rightarrow \mathbb{R}\) be a strictly decreasing continuous function such that \(\lim_{x \to \infty} f(x) = 0\). Prove that

\[
\int_0^\infty \frac{f(x) - f(x+1)}{f(x)} \, dx
\]
diverges.

II. (10 MINUTES): DISCUSSION

Discuss whatever ideas you had, and to name one of two problems that you think are most likely to be solvable. It is OK if
you do not have any solutions at this point. Our goal here is learning a skill: how to pick out the one or two problems that you
might devote significant time towards solving. Just getting credit for one solution is already an accomplishment!

III. (10 MINUTES): WRITE UP

If you have a solution, try to write it up in acceptable form. Since this is a practice situation, it is alright to ask about points of
style. In general you can refer to theorems of the math curriculum without proof, but it helps if you know their names, and you
state them correctly.

The next page has hints, don’t turn the page until you want to see them.
You won’t get these on a real exam, but below are a few ideas that might help. Look these over, and see if they help you make further progress on any of these.

A–1 Nothing fancy here. Argue that the common sum is at least $n$. Maybe consider the odd and even cases separately.

A–2 There is an “obvious” family of functions $f$ where this is true. Look at properties of the function $f(x + 1) - f(x)$.

A–3 If $a$ and $b$ are both zero, this is easy. Otherwise, look at the behavior of $h$ along a line in the direction $(a, b)$.

A–4 Look for a divisor of the form $10^j + 1$. You might first suppose that $n$ is odd.

A–5 (Hard!) As a first step, suppose $x, y \in G$ are orthogonal, and look at $x \ast (x \ast y) = (x \ast x) \ast y$.

A–6 (Hard!) For an integer $a$, argue there is a large enough integer $b$, so that the integral from $a$ to $b$ is at least $1/2$. Or: look at whether $f(x + 1)/f(x)$ tends to zero, for large $x$.

_WARNING: the next page has full solutions, don’t turn the page until you want to see them!
V. FULL SOLUTIONS

A–1 The largest such $k$ is $\lfloor \frac{n+1}{2} \rfloor = \lfloor \frac{n}{2} \rfloor$. For $n$ even, this value is achieved by the partition

\{1,n\}, \{2,n-1\}, \ldots ;

for $n$ odd, it is achieved by the partition

\{n\}, \{1,n-1\}, \{2,n-2\}, \ldots .

One way to see that this is optimal is to note that the common sum can never be less than $n$, since $n$ itself belongs to one of the boxes. This implies that $k \leq (1 + \cdots + n)/n = (n+1)/2$. Another argument is that if $k > (n+1)/2$, then there would have to be two boxes with one number each (by the pigeonhole principle), but such boxes could not have the same sum.

**Remark.** A much subtler question would be to find the smallest $k$ (as a function of $n$) for which no such arrangement exists.

A–2 The only such functions are those of the form $f(x) = cx + d$ for some real numbers $c, d$ (for which the property is obviously satisfied). To see this, suppose that $f$ has the desired property. Then for any $x \in \mathbb{R}$,

$$2f'(x) = f(x+2) - f(x) = (f(x+2) - f(x+1)) + (f(x+1) - f(x)) = f'((x+1)) + f'(x).$$

Consequently, $f'(x+1) = f'(x)$.

Define the function $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = f(x+1) - f(x)$, and put $c = g(0), d = f(0)$. For all $x \in \mathbb{R}$, $g'(x) = f'(x+1) - f'(x) = 0$, so $g(x) = c$ identically, and $f'(x) = f(x+1) - f(x) = g(x) = c$, so $f(x) = cx + d$ identically as desired.

A–3 If $a = b = 0$, then the desired result holds trivially, so we assume that at least one of $a, b$ is nonzero. Pick any point $(a_0, b_0) \in \mathbb{R}^2$, and let $L$ be the line given by the parametric equation $L(t) = (a_0, b_0) + (a, b)t$ for $t \in \mathbb{R}$. By the chain rule and the given equation, we have $\frac{d}{dt}(h \circ L) = h \circ L$. If we write $f = h \circ L : \mathbb{R} \to \mathbb{R}$, then $f'(t) = f(t)$ for all $t$. It follows that $f(t) = Ce^t$ for some constant $C$. Since $|f(t)| \leq M$ for all $t$, we must have $C = 0$. It follows that $h(a_0, b_0) = 0$; since $(a_0, b_0)$ was an arbitrary point, $h$ is identically 0 over all of $\mathbb{R}^2$.

A–4 Put

$$N = 10^{10^{10^9}} + 10^{10^9} + 10^8 - 1.$$

Write $n = 2^mk$ with $m$ a nonnegative integer and $k$ a positive odd integer. For any nonnegative integer $j$,

$$10^{2^mj} \equiv (-1)^j \pmod{10^{2^m} + 1}.$$

Since $10^n \geq n \geq 2^m \geq m + 1$, $10^n$ is divisible by $2^n$ and hence by $2^{m+1}$, and similarly $10^{10^9}$ is divisible by $2^{10^9}$ and hence by $2^{m+1}$. It follows that

$$N \equiv 1 + 1 + (-1) + (-1) \equiv 0 \pmod{10^{2^n} + 1}.$$

Since $N \geq 10^{10^{10^9}} > 10^n + 1 \geq 10^{2^m} + 1$, it follows that $N$ is composite.

A–5 We start with three lemmas.

**Lemma 1.** If $\mathbf{x}, \mathbf{y} \in G$ are nonzero orthogonal vectors, then $\mathbf{x} \times \mathbf{x}$ is parallel to $\mathbf{y}$.

**Proof.** Put $\mathbf{z} = \mathbf{x} \times \mathbf{y} \neq 0$, so that $\mathbf{x}, \mathbf{y}$, and $\mathbf{z} = \mathbf{x} \times \mathbf{y}$ are nonzero and mutually orthogonal. Then $\mathbf{w} = \mathbf{x} \times \mathbf{z} \neq 0$, so $\mathbf{w} = \mathbf{x} \times \mathbf{z}$ is nonzero and orthogonal to $\mathbf{x}$ and $\mathbf{z}$. However, if $(\mathbf{x} \times \mathbf{z}) \times \mathbf{y} \neq 0$, then $\mathbf{w} = \mathbf{x} \times (\mathbf{x} \times \mathbf{y}) = (\mathbf{x} \times \mathbf{y}) \times \mathbf{y} = (\mathbf{x} \times \mathbf{x}) \times \mathbf{y}$ is also orthogonal to $\mathbf{y}$, a contradiction. \hfill $\square$

**Lemma 2.** If $\mathbf{x} \in G$ is nonzero, and there exists $\mathbf{y} \in G$ nonzero and orthogonal to $\mathbf{x}$, then $\mathbf{x} \times \mathbf{x} = 0$.

**Proof.** Lemma 1 implies that $\mathbf{x} \times \mathbf{x}$ is parallel to both $\mathbf{y}$ and $\mathbf{x} \times \mathbf{y}$, so it must be zero. \hfill $\square$

**Lemma 3.** If $\mathbf{x}, \mathbf{y} \in G$ commute, then $\mathbf{x} \times \mathbf{y} = 0$.
Proof. If $x \times y \neq 0$, then $y \times x$ is nonzero and distinct from $x \times y$. Consequently, $x \times y = x \times y$ and $y \times x = y \times x \neq x \times y$. □

We proceed now to the proof. Assume by way of contradiction that there exist $a, b \in G$ with $a \times b \neq 0$. Put $c = a \times b = a \star b$, so that $a, b, c$ are nonzero and linearly independent. Let $e$ be the identity element of $G$. Since $e$ commutes with $a, b, c$, by Lemma 3 we have $e \times a = e \times b = e \times c = 0$. Since $a, b, c$ span $\mathbb{R}^3$, $e \times x = 0$ for all $x \in \mathbb{R}^3$, so $e = 0$.

Since $b, c$, and $b \times c = b \star c$ are nonzero and mutually orthogonal, Lemma 2 implies

$$b \times b = c \times c = (b \star c) \star (b \star c) = 0 = e.$$  

Hence $b \star c = e \star b$, contradicting Lemma 3 because $b \times c \neq 0$. The desired result follows.

A–6 First solution skipped, too hard.

Second solution. (Communicated by Paul Allen.) Let $b > a$ be nonnegative integers. Then

$$
\int_a^b \frac{f(x) - f(x+1)}{f(x)} \, dx = \sum_{k=a}^{b-1} \int_0^1 \frac{f(x+k) - f(x+k+1)}{f(x+k)} \, dx
$$

$$= \int_0^1 \sum_{k=a}^{b-1} \frac{f(x+k) - f(x+k+1)}{f(x+k)} \, dx
$$

$$\geq \int_0^1 \sum_{k=a}^{b-1} \frac{f(x+k) - f(x+k+1)}{f(x+a)} \, dx
$$

$$= \int_0^1 \frac{f(x+a) - f(x+b)}{f(x+a)} \, dx.
$$

Now since $f(x) \to 0$, given $a$, we can choose an integer $l(a) > a$ for which $f(l(a)) < f(a+1)/2$; then $\frac{f(x+a) - f(x+l(a))}{f(x+a)} \geq 1 - \frac{f(l(a))}{f(a+1)} > 1/2$ for all $x \in [0, 1]$. Thus if we define a sequence of integers $a_n$ by $a_0 = 0$, $a_{n+1} = l(a_n)$, then

$$
\int_0^\infty \frac{f(x) - f(x+1)}{f(x)} \, dx = \sum_{n=0}^\infty \int_{a_n}^{a_{n+1}} \frac{f(x) - f(x+1)}{f(x)} \, dx
$$

$$> \sum_{n=0}^\infty \int_0^1 (1/2) \, dx,
$$

and the final sum clearly diverges.

Third solution. (By Joshua Rosenberg, communicated by Catalin Zara.) If the original integral converges, then on one hand the integrand $(f(x) - f(x+1))/f(x) = 1 - f(x+1)/f(x)$ cannot tend to 1 as $x \to \infty$. On the other hand, for any $a \geq 0$,

$$0 < \frac{f(a+1)}{f(a)}
$$

$$< \frac{1}{f(a)} \int_a^{a+1} f(x) \, dx
$$

$$= \frac{1}{f(a)} \int_0^\infty (f(x) - f(x+1)) \, dx
$$

$$\leq \int_a^\infty \frac{f(x) - f(x+1)}{f(x)} \, dx,
$$

and the last expression tends to 0 as $a \to \infty$. Hence by the squeeze theorem, $f(a+1)/f(a) \to 0$ as $a \to \infty$, a contradiction.