

Combinatorial Putnam Review Problems

(as selected by Andrzej Rucinski)

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1985 A–1 Determine, with proof, the number of ordered triples (A_1, A_2, A_3) of sets which have the property that

- (i) $A_1 \cup A_2 \cup A_3 = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$,
and
- (ii) $A_1 \cap A_2 \cap A_3 = \emptyset$.

Express your answer in the form $2^a 3^b 5^c 7^d$, where a, b, c, d are nonnegative integers.

1988 A–4 (a) If every point of the plane is painted one of three colors, do there necessarily exist two points of the same color exactly one inch apart?

(b) What if “three” is replaced by “nine”?

1990 A–3* Prove that any convex pentagon whose vertices (no three of which are collinear) have integer coordinates must have area greater than or equal to $5/2$.

1992 B–1 Let S be a set of n distinct real numbers. Let A_S be the set of numbers that occur as averages of two distinct elements of S . For a given $n \geq 2$, what is the smallest possible number of elements in A_S ?

1993 A–3* Let \mathcal{P}_n be the set of subsets of $\{1, 2, \dots, n\}$. Let $c(n, m)$ be the number of functions $f : \mathcal{P}_n \rightarrow \{1, 2, \dots, m\}$ such that $f(A \cap B) = \min\{f(A), f(B)\}$. Prove that

$$c(n, m) = \sum_{j=1}^m j^n.$$

1993 B–2 Consider the following game played with a deck of $2n$ cards numbered from 1 to $2n$. The deck is randomly shuffled and n cards are dealt to each

of two players. Beginning with A , the players take turns discarding one of their remaining cards and announcing its number. The game ends as soon as the sum of the numbers on the discarded cards is divisible by $2n + 1$. The last person to discard wins the game. Assuming optimal strategy by both A and B , what is the probability that A wins?

1994 A–3 Show that if the points of an isosceles right triangle of side length 1 are each colored with one of four colors, then there must be two points of the same color which are at least a distance $2 - \sqrt{2}$ apart.

1995 B–1 For a partition π of $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, let $\pi(x)$ be the number of elements in the part containing x . Prove that for any two partitions π and π' , there are two distinct numbers x and y in $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ such that $\pi(x) = \pi(y)$ and $\pi'(x) = \pi'(y)$. [A *partition* of a set S is a collection of disjoint subsets (parts) whose union is S .]

1996 A–3 Suppose that each of 20 students has made a choice of anywhere from 0 to 6 courses from a total of 6 courses offered. Prove or disprove: there are 5 students and 2 courses such that all 5 have chosen both courses or all 5 have chosen neither course.

1996 B–1* Define a **selfish** set to be a set which has its own cardinality (number of elements) as an element. Find, with proof, the number of subsets of $\{1, 2, \dots, n\}$ which are *minimal* selfish sets, that is, selfish sets none of whose proper subsets is selfish.

2000 B–6 Let B be a set of more than $2^{n+1}/n$ distinct points with coordinates of the form $(\pm 1, \pm 1, \dots, \pm 1)$ in n -dimensional space with $n \geq 3$. Show that there are three distinct points in B which are the vertices of an equilateral triangle.