Easy or Hard? What would you try?

December 2, 2005

Let $f(x)$ be a continuous real-valued function defined on the interval $[0, 1]$. Show that

$$\int_0^1 \int_0^1 |f(x) + f(y)|\, dx\, dy \geq \int_0^1 |f(x)|\, dx.$$

let H be the unit hemisphere $\{(x, y, z) : x^2 + y^2 + z^2 = 1, z \geq 0\}$, C the unit circle $\{(x, y, 0) : x^2 + y^2 = 1\}$, and P the regular pentagon inscribed in C. Determine the surface area of that portion of H lying over the planar region inside P, and write your answer in the form $A \sin \alpha + B \cos \beta$, where A, B, α, β are real numbers.

A game starts with four heaps of beans, containing 3,4,5 and 6 beans. The two players move alternately. A move consists of taking **either**
a) one bean from a heap, provided at least two beans are left in that heap, **or**
b) a complete heap of two or three beans.
The player who takes the last heap wins. To win the game, do you want to move first or second? Give a winning strategy.

Do there exist polynomials $a(x), b(x), c(y), d(y)$ such that $1+xy+x^2y^2 = a(x)c(y)+b(x)d(y)$ holds identically?

An absentminded professor buys two boxes of matches and puts them in his pocket. Every time he needs a match, he selects at random (with equal probability) from one or other of the boxes. One day the professor opens a matchbox and finds that it is empty. (He must have absentmindedly put the empty box back in his pocket when he took the last match from it.) If each box originally contained n matches, what is the probability that the other box currently contains k matches (for each k, $0 \leq k \leq n$)?