These are solutions to the "power series" problems distributed by email on Monday, 11/17/03. — MG

1992 B-2: For nonnegative integers n and k, define $Q(n,k)$ to be the coefficient of x^k in the expansion of $(1+x+x^2+x^3)^n$. Prove that

$$Q(n,k) = \sum_{j=0}^{k} \binom{n}{j}\binom{n}{k-2j},$$

where $\binom{a}{b}$ is the standard binomial coefficient. (Reminder: For integers a and b with $a \geq 0$, $\binom{a}{b} = \frac{a!}{b!(a-b)!}$ for $0 \leq b \leq a$, with $\binom{a}{b} = 0$ otherwise.)

Solution: $(1+x+x^2+x^3)^n = (1+x)^n(1+x^2)^n = (\sum_{i\geq 0}\binom{n}{i}x^i)(\sum_{j\geq 0}\binom{n}{j}x^{2j})$. We need no explicit upper bounds on i and j, since the binomial coefficients are zero for i or j greater than n. Then the coefficient $Q(n,k)$ of x^k is the sum of $\binom{n}{i}\binom{n}{j}$ over all pairs (i,j) such that $i \geq 0$, $j \geq 0$, and $i+2j=k$.

1999 A-3: Consider the power series expansion

$$\frac{1}{1-2x-x^2} = \sum_{n=0}^{\infty} a_n x^n.$$

Prove that, for each integer $n \geq 0$, there is an integer m such that

$$a_n^2 + a_{n+1}^2 = a_m.$$

Solution: Computing the coefficient of x^{n+1} in $(1-2x-x^2)\sum_{m=0}^{\infty} a_m x^m = 1$ yields the recurrence $a_{n+1} = 2a_n + a_{n-1}$; the sequence $\{a_n\}$ is then characterized by this recurrence and the initial conditions $a_0 = 1, a_1 = 2$.

Define the sequence $\{b_n\}$ by $b_{2n} = a_{n-1}^2 + a_n^2,\ b_{2n+1} = a_n(a_{n-1} + a_{n+1})$. Then

$$\begin{aligned} 2b_{2n+1} + b_{2n} &= 2a_n a_{n+1} + 2a_{n-1}a_n + a_{n-1}^2 + a_n^2 \\ &= 2a_n a_{n+1} + a_{n-1}a_{n+1} + a_n^2 \\ &= a_{n+1}^2 + a_n^2 = b_{2n+2}, \end{aligned}$$

and similarly $2b_{2n} + b_{2n-1} = b_{2n+1}$, so that $\{b_n\}$ satisfies the same recurrence as $\{a_n\}$. Since further $b_0 = 1, b_1 = 2$ (where we use the recurrence for $\{a_n\}$ to calculate $a_{-1} = 0$), we deduce that $b_n = a_n$ for all n. In particular, $a_n^2 + a_{n+1}^2 = b_{2n+2} = a_{2n+2}$.

Second solution: Note that

$$\frac{1}{1-2x-x^2} = \frac{1}{2\sqrt{2}}\left(\frac{\sqrt{2}+1}{1-(1+\sqrt{2})x} + \frac{\sqrt{2}-1}{1-(1-\sqrt{2})x}\right)$$

and that

$$\frac{1}{1+(1\pm\sqrt{2})x} = \sum_{n=0}^{\infty}(1\pm\sqrt{2})^n x^n,$$

so that

$$a_n = \frac{1}{2\sqrt{2}}\left((\sqrt{2}+1)^{n+1} - (1-\sqrt{2})^{n+1}\right).$$

A simple computation (omitted here) now shows that $a_n^2 + a_{n+1}^2 = a_{2n+2}$.

Third solution (by Richard Stanley): Let A be the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$. A simple induction argument shows that

$$A^{n+2} = \begin{pmatrix} a_n & a_{n+1} \\ a_{n+1} & a_{n+2} \end{pmatrix}.$$

The desired result now follows from comparing the top left corner entries of the equality $A^{n+2}A^{n+2} = A^{2n+4}$.

1999 B-3: Let $A = \{(x,y) : 0 \le x, y < 1\}$. For $(x,y) \in A$, let

$$S(x,y) = \sum_{\frac{1}{2} \le \frac{m}{n} \le 2} x^m y^n,$$

where the sum ranges over all pairs (m,n) of positive integers satisfying the indicated inequalities. Evaluate

$$\lim_{(x,y)\to(1,1),(x,y)\in A} (1-xy^2)(1-x^2y)S(x,y).$$

Solution: We first note that

$$\sum_{m,n>0} x^m y^n = \frac{xy}{(1-x)(1-y)}.$$

Subtracting S from this gives two sums, one of which is

$$\sum_{m\ge 2n+1} x^m y^n = \sum_n y^n \frac{x^{2n+1}}{1-x} = \frac{x^3y}{(1-x)(1-x^2y)}$$

and the other of which sums to $xy^3/[(1-y)(1-xy^2)]$. Therefore

$$\begin{aligned} S(x,y) &= \frac{xy}{(1-x)(1-y)} - \frac{x^3y}{(1-x)(1-x^2y)} - \frac{xy^3}{(1-y)(1-xy^2)} \\ &= \frac{xy(1+x+y+xy-x^2y^2)}{(1-x^2y)(1-xy^2)} \end{aligned}$$

and the desired limit is $\lim_{(x,y)\to(1,1)} xy(1+x+y+xy-x^2y^2) = 3$.