

# The 55th William Lowell Putnam Mathematical Competition

## Saturday, December 3, 1994

October 6, 2000

- A-1 Let  $(a_n)$  be a sequence of positive reals such that, for all  $n$ ,  $a_n \leq a_{2n} + a_{2n+1}$ . Prove that  $\sum_{n=1}^{\infty} a_n$  diverges.
- A-2 Find the positive value of  $m$  such that the area in the first quadrant enclosed by the ellipse  $\frac{x^2}{9} + y^2 = 1$ , the  $x$ -axis, and the line  $y = 2x/3$  is equal to the area in the first quadrant enclosed by the ellipse  $\frac{x^2}{9} + y^2 = 1$ , the  $y$ -axis, and the line  $y = mx$ .
- A-3 Prove that the points of an isosceles triangle of side length 1 cannot be colored in four colors such that no two points at distance at least  $2 - \sqrt{2}$  from each other receive the same color.
- A-4 Let  $A$  and  $B$  be  $2 \times 2$  matrices with integer entries such that each of  $A$ ,  $A + B$ ,  $A + 2B$ ,  $A + 3B$ ,  $A + 4B$  has an inverse with integer entries. Prove that the same must be true of  $A + 5B$ .
- A-5 Let  $(r_n)$  be a sequence of positive reals with limit 0. Let  $S$  be the set of all numbers expressible in the form  $r_{i_1} + \dots + r_{i_{1994}}$  for positive integers  $i_1 < i_2 < \dots < i_{1994}$ . Prove that every interval  $(a, b)$  contains a subinterval  $(c, d)$  whose intersection with  $S$  is empty.
- A-6 Let  $f_1, \dots, f_{10}$  be bijections of the integers such that for every integer  $n$ , there exists a sequence  $i_1, \dots, i_k$  for some  $k$  such that  $f_{i_1} \circ \dots \circ f_{i_k}(0) = n$ . Prove that if  $A$  is any nonempty finite set, there exist at most 512 sequences  $(e_1, \dots, e_{10})$  of zeroes and ones such that  $f_1^{e_1} \circ \dots \circ f_{10}^{e_{10}}$  maps  $A$  to  $A$ . (Here  $f_1^1 = f$  and  $f^0$  means the identity function.)
- B-1 Find all positive integers  $n$  such that  $|n - m^2| \leq 250$  for exactly 15 nonnegative integers  $m$ .
- B-2 Find all  $c$  such that the graph of the function  $x^4 + 9x^3 + cx^2 + ax + b$  meets some line in four distinct points.
- B-3 Let  $f(x)$  be a positive-valued function over the reals such that  $f'(x) > f(x)$  for all  $x$ . For what  $k$  must there exist  $N$  such that  $f(x) > e^{kx}$  for  $x > N$ ?
- B-4 Let  $A$  be the matrix  $\begin{pmatrix} 3 & 2 \\ 2 & 4 \end{pmatrix}$  and for positive integers  $n$ , define  $d_n$  as the greatest common divisor of the entries of  $A^n - I$ , where  $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ . Prove that  $d_n \rightarrow \infty$  as  $n \rightarrow \infty$ .
- B-5 Fix  $n$  a positive integer. For  $\alpha$  real, define  $f_\alpha(i)$  as the greatest integer less than or equal to  $\alpha i$ , and write  $f^k$  for the  $k$ -th iterate of  $f$  (i.e.  $f^1 = f$  and  $f^{k+1} = f \circ f^k$ ). Prove there exists  $\alpha$  such that  $f_{\alpha^k}(n^2) = f_\alpha^k(n^2) = n^2 - k$  for  $k = 1, \dots, n$ .
- B-6 Suppose  $a, b, c, d$  are integers with  $0 \leq a \leq 99$ ,  $0 \leq c \leq d \leq 99$ . For any integer  $i$ , let  $n_i = 101i + 1002^i$ . Show that if  $n_a + n_b$  is congruent to  $n_c + n_d \pmod{10100}$ , then  $a = c$  and  $b = d$ .