

# The 59th William Lowell Putnam Mathematical Competition

## Saturday, December 5, 1998

October 6, 2000

A-1 A right circular cone has base of radius 1 and height 3. A cube is inscribed in the cone so that one face of the cube is contained in the base of the cone. What is the side-length of the cube?

A-2 Let  $s$  be any arc of the unit circle lying entirely in the first quadrant. Let  $A$  be the area of the region lying below  $s$  and above the  $x$ -axis and let  $B$  be the area of the region lying to the right of the  $y$ -axis and to the left of  $s$ . Prove that  $A + B$  depends only on the arc length, and not on the position, of  $s$ .

A-3 Let  $f$  be a real function on the real line with continuous third derivative. Prove that there exists a point  $a$  such that

$$f(a) \cdot f'(a) \cdot f''(a) \cdot f'''(a) \geq 0.$$

A-4 Let  $A_1 = 0$  and  $A_2 = 1$ . For  $n > 2$ , the number  $A_n$  is defined by concatenating the decimal expansions of  $A_{n-1}$  and  $A_{n-2}$  from left to right. For example  $A_3 = A_2A_1 = 10$ ,  $A_4 = A_3A_2 = 101$ ,  $A_5 = A_4A_3 = 10110$ , and so forth. Determine all  $n$  such that 11 divides  $A_n$ .

A-5 Let  $\mathcal{F}$  be a finite collection of open discs in  $\mathbb{R}^2$  whose union contains a set  $E \subseteq \mathbb{R}^2$ . Show that there is a pairwise disjoint subcollection  $D_1, \dots, D_n$  in  $\mathcal{F}$  such that

$$E \subseteq \cup_{j=1}^n 3D_j.$$

Here, if  $D$  is the disc of radius  $r$  and center  $P$ , then  $3D$  is the disc of radius  $3r$  and center  $P$ .

A-6 Let  $A, B, C$  denote distinct points with integer coordinates in  $\mathbb{R}^2$ . Prove that if

$$(|AB| + |BC|)^2 < 8 \cdot [ABC] + 1$$

then  $A, B, C$  are three vertices of a square. Here  $|XY|$  is the length of segment  $XY$  and  $[ABC]$  is the area of triangle  $ABC$ .

B-1 Find the minimum value of

$$\frac{(x + 1/x)^6 - (x^6 + 1/x^6) - 2}{(x + 1/x)^3 + (x^3 + 1/x^3)}$$

for  $x > 0$ .

B-2 Given a point  $(a, b)$  with  $0 < b < a$ , determine the minimum perimeter of a triangle with one vertex at  $(a, b)$ , one on the  $x$ -axis, and one on the line  $y = x$ . You may assume that a triangle of minimum perimeter exists.

B-3 Let  $H$  be the unit hemisphere  $\{(x, y, z) : x^2 + y^2 + z^2 = 1, z \geq 0\}$ ,  $C$  the unit circle  $\{(x, y, 0) : x^2 + y^2 = 1\}$ , and  $P$  the regular pentagon inscribed in  $C$ . Determine the surface area of that portion of  $H$  lying over the planar region inside  $P$ , and write your answer in the form  $A \sin \alpha + B \cos \beta$ , where  $A, B, \alpha, \beta$  are real numbers.

B-4 Find necessary and sufficient conditions on positive integers  $m$  and  $n$  so that

$$\sum_{i=0}^{mn-1} (-1)^{\lfloor i/m \rfloor + \lfloor i/n \rfloor} = 0.$$

B-5 Let  $N$  be the positive integer with 1998 decimal digits, all of them 1; that is,

$$N = 1111 \cdots 11.$$

Find the thousandth digit after the decimal point of  $\sqrt{N}$ .

B-6 Prove that, for any integers  $a, b, c$ , there exists a positive integer  $n$  such that  $\sqrt{n^3 + an^2 + bn + c}$  is not an integer.