

# The 61st William Lowell Putnam Mathematical Competition

## Saturday, December 2, 2000

November 8, 2001

A-1 Let  $A$  be a positive real number. What are the possible values of  $\sum_{j=0}^{\infty} x_j^2$ , given that  $x_0, x_1, \dots$  are positive numbers for which  $\sum_{j=0}^{\infty} x_j = A$ ?

A-2 Prove that there exist infinitely many integers  $n$  such that  $n, n+1, n+2$  are each the sum of the squares of two integers. [Example:  $0 = 0^2 + 0^2, 1 = 0^2 + 1^2, 2 = 1^2 + 1^2$ .]

A-3 The octagon  $P_1P_2P_3P_4P_5P_6P_7P_8$  is inscribed in a circle, with the vertices around the circumference in the given order. Given that the polygon  $P_1P_3P_5P_7$  is a square of area 5, and the polygon  $P_2P_4P_6P_8$  is a rectangle of area 4, find the maximum possible area of the octagon.

A-4 Show that the improper integral

$$\lim_{B \rightarrow \infty} \int_0^B \sin(x) \sin(x^2) dx$$

converges.

A-5 Three distinct points with integer coordinates lie in the plane on a circle of radius  $r > 0$ . Show that two of these points are separated by a distance of at least  $r^{1/3}$ .

A-6 Let  $f(x)$  be a polynomial with integer coefficients. Define a sequence  $a_0, a_1, \dots$  of integers such that  $a_0 = 0$  and  $a_{n+1} = f(a_n)$  for all  $n \geq 0$ . Prove that if there exists a positive integer  $m$  for which  $a_m = 0$  then either  $a_1 = 0$  or  $a_2 = 0$ .

B-1 Let  $a_j, b_j, c_j$  be integers for  $1 \leq j \leq N$ . Assume for each  $j$ , at least one of  $a_j, b_j, c_j$  is odd. Show that

there exist integers  $r, s, t$  such that  $ra_j + sb_j + tc_j$  is odd for at least  $4N/7$  values of  $j, 1 \leq j \leq N$ .

B-2 Prove that the expression

$$\frac{\gcd(m, n)}{n} \binom{n}{m}$$

is an integer for all pairs of integers  $n \geq m \geq 1$ .

B-3 Let  $f(t) = \sum_{j=1}^N a_j \sin(2\pi jt)$ , where each  $a_j$  is real and  $a_N$  is not equal to 0. Let  $N_k$  denote the number of zeroes (including multiplicities) of  $\frac{d^k f}{dt^k}$ . Prove that

$$N_0 \leq N_1 \leq N_2 \leq \dots \text{ and } \lim_{k \rightarrow \infty} N_k = 2N.$$

[Editorial clarification: only zeroes in  $[0, 1)$  should be counted.]

B-4 Let  $f(x)$  be a continuous function such that  $f(2x^2 - 1) = 2xf(x)$  for all  $x$ . Show that  $f(x) = 0$  for  $-1 \leq x \leq 1$ .

B-5 Let  $S_0$  be a finite set of positive integers. We define finite sets  $S_1, S_2, \dots$  of positive integers as follows: the integer  $a$  is in  $S_{n+1}$  if and only if exactly one of  $a-1$  or  $a$  is in  $S_n$ . Show that there exist infinitely many integers  $N$  for which  $S_N = S_0 \cup \{N+a : a \in S_0\}$ .

B-6 Let  $B$  be a set of more than  $2^{n+1}/n$  distinct points with coordinates of the form  $(\pm 1, \pm 1, \dots, \pm 1)$  in  $n$ -dimensional space with  $n \geq 3$ . Show that there are three distinct points in  $B$  which are the vertices of an equilateral triangle.